FOR ERRATA

AD 262056

THE FOLLOWING PAGES ARE CHANGES

TO BASIC DOCUMENT

AD 262056

U. S. NAVAL ORDNANCE TEST STATION
CHINA LAKE, CALIFORNIA

IN REPLY REFER TO:
751/LRC:mm
25 September 1963

From: Commander, U. S. Naval Ordnance Test Station
To: Initial Distribution List of NAVWEPS Report 7741

Subj: NAVWEPS Report 7741, Determination of Rocket-Motor Mass by Measurement of the Natural Frequency of a Mass-Spring System, dated 15 June 1961; transmittal of errata sheets for

Encl: (1) Page 15 for subject report
(2) Page 29 for subject report
(3) Page 47 for subject report

1. It is requested that pages 15, 29, and 47 of the subject report be replaced with the attached new pages 15, 29, and 47, which have been printed on gummed paper for your convenience.

2. On page 25 of the subject report, change Eq. 53 to read as follows:

$$\varphi = 1 - e^{-t\xi\omega_2}$$

and change the line of text that follows Eq. 53 to read, '' where φ is the phase shift and $\frac{1}{\xi\omega_2}$ is the time constant.''

3. On page 30, change paragraph 4 to read, ''To keep the period accuracy to within 190 μsec, the zero crossover points of the 16-cps signal must be held to half the 190 μsec.'' Change Eq. 68 to read as follows:

$$\Delta E = 102\frac{\Delta t}{2} \text{ volts} = \frac{(102)(190)10^{-6}}{2} \text{ volts} = .0096 \text{ volts}$$

And change the first sentence of the last paragraph on page 30 to read, ''Therefore, the 1-cps signal must be reduced by the ratio $\frac{.0096}{1.15} = .0084$ or by -41.5 db.''

4. On page 53, paragraph 4, line 3, delete ''more than''; on line 4, delete ''31 db'' and substitute ''41.5 db.''

5. On page 59, paragraph 1, sentence 2, delete ''1.5°'' and substitute ''1.7°''; delete the words ''if the damping of the system were twice that which was estimated''; and change ''.5 percent'' to ''.6 percent.''

C. E. Van Hagan
C. E. VAN HAGAN
By direction

where ω_1 is the driving frequency,

ω is the natural frequency of the system,

ξ is the damping ratio,

Ψ is the phase lag between the driving function and the spring-mass system.

To maintain the overall accuracy of the mass-measuring system to 1 percent, we will attempt to keep the phase lag Ψ to a value less than that which would produce an error somewhat less than 1 percent. As will be shown later, a value of phase lag corresponding to a .6 percent mass error can be achieved.

Now consider an ω_1

$$\omega_1 = \sqrt{\frac{K}{M}} \tag{32}$$

and an ω

$$\omega = \sqrt{\frac{K}{1.006M}} = \frac{1}{1.003}\sqrt{\frac{K}{M}} \tag{33}$$

and therefore

$$\omega_1 = 1.003\omega. \tag{34}$$

Then, substituting in Equation (6),

$$\Psi = \tan^{-1} \frac{2\xi(1.003)\omega^2}{\omega^2 - (1.006)\omega^2} = \tan^{-1} \frac{2\xi(1.003)}{-(.006)} \tag{35}$$

For a damping ratio of .15, in the spring-mass system, $\Psi = 91.5°$ and therefore the servo system must follow to within 1.5°.

For a damping ratio of .1°, $\Psi = 91.75°$ and therefore the servo system must follow to within 1.75°.

exactly perpendicular to the vertical, and that in fact the thrust axis is likely to wander in small angles about the assumed axis. Also, the motor with which we are concerned in this paper has thrust deflectors (jetavators) that produce a 10,000-lb side thrust. The cycling rate of the deflector can be as high as 1 cps.

From the equation $\omega_1 = 1.003\ \omega$, for a mass measurement of 0.6 percent accuracy, the frequency must be determined to within .3 percent.

The period, T, of a sine wave is

(61) $$T = \frac{2\pi}{\omega}$$

and

(62) $$dT = \frac{-2\pi}{\omega^2}\, d\omega \ .$$

For small differences the equation becomes

(63) $$\Delta T = \frac{-2\pi}{\omega^2}\Delta\omega \quad .$$

For $\Delta\omega = .003\ \omega$, and $\omega = 102$ rad/sec, ΔT becomes

(64) $$\Delta T = \frac{-2\pi\ (.003)}{102} = 190\ \mu\text{sec} \quad .$$

Consider the system when the resonant frequency is 102 rad/sec. The force supplied by the eccentric mass is then

(65) $f(t) = mr\omega^2 \sin \omega t = (.187)(102)^2 \sin 102t = 1950 \sin 102t$.

Let the voltage amplitude, E, derived from the transducer detecting the amplitude of the motor's vertical motion, be one volt.

(89) $$e_{ss}(nT)_{10} = \lim_{z \to 1} \frac{z-1}{z} (.35)\frac{1}{2} (.05)^2 \frac{z(z+1)}{(z-1)^3}$$

$$\left[\frac{1}{1 + \frac{(20.2)z}{121} \frac{(4.75)(z - .956)}{(z-1)^2 (z - .58)}} \right] = .01 \text{ radian}$$

$$= .57 \text{ degrees.}$$

The steady-state error for the end-of-burning case (system oscillating at 16 cps) is then

(90) $$e_{ss}(nT)_{16} = \lim_{z \to 1} \frac{z-1}{z} \frac{1}{2} (.031)^2 \frac{z(z+1)}{(z-1)^3}$$

$$\left[\frac{1}{1 + \frac{12.6}{121} \frac{(3.13)z\ (z - .974)}{(z-1)^2 (z - .72)}} \right] = .03 \text{ radian}$$

$$= 1.7 \text{ degrees.}$$

It is seen that these steady-state errors, when compared to those given on the bottom of page 15, are adequate for the required system accuracy.

RESPONSE TO A SUDDEN MASS CHANGE

Given the open-loop z-transfer function of the control system for the before-burning case

(91) $$G(z) = \frac{20(4.75\ z)(z - .956)}{121\ (z-1)^2 (z - .58)}$$

the error function is given by

(92) $$E(z) = \frac{R(z)}{1 + G(z)} = \frac{R(z)}{1 + \frac{20(4.75\ z)(z - .956)}{121\ (z-1)^2 (z - .58)}}$$

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DETERMINATION OF ROCKET-MOTOR MASS BY MEASUREMENT OF THE NATURAL FREQUENCY OF A MASS-SPRING SYSTEM

By
Benjamin Glatt
Test Department

ABSTRACT. A system is described whereby the mass of a rocket motor being tested in a static test stand can be determined continuously while burning. The motor is mounted on springs and driven by the force of a rotating eccentric mass so as to oscillate at the natural frequency of the mass and spring system. Given the spring constant, the mass of the system can then be calculated from the frequency of oscillation. Given a required accuracy of 1 percent for the measurement of the mass of the motor, the specifications for the control system are determined.

A control system is designed, analyzed, and shown to meet the requirements of the system with the expected input signals and external disturbances.

U.S. NAVAL ORDNANCE TEST STATION
China Lake, California
15 June 1961

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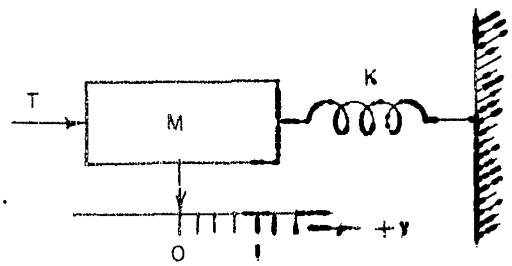
CONTENTS

FIGURES

INTRODUCTION

BACKGROUND

The evaluation of large solid-propellant rocket motors requires a knowledge of the thrust of the motor throughout its burning period. A model of a thrust stand is shown in Fig. 1.



$$\Sigma F_y = M\ddot{y} + Ky - T = 0$$

where T = Thrust
K = Spring constant
M(t) = Mass as a function of time
$\ddot{y}$ = Acceleration

Fig. 1. Schematic Diagram of Conventional Rocket Test Stand.

Conventionally, static test stands are equipped with load cells that measure the displacement, y, and only the static information is obtained; i.e., the equation $F_y = Ky - T = 0$ is the equation used to determine the thrust. However, to obtain instantaneous specific impulse and high-frequency thrust measurements, the acceleration term must be considered.

The acceleration can be obtained by placing an accelerometer on the rocket motor to measure the acceleration along the axis of thrust. It then remains to determine the mass of the motor at any time during the period of burning. A method to measure continuously the mass of a rocket motor while burning in a test stand is the subject of this paper.

STATEMENT OF PROBLEM

Consider a system whereby a mass (say, of a rocket motor) is mounted on linear springs, and mounted on this mass is a rotating eccentric mass (Fig. 2).

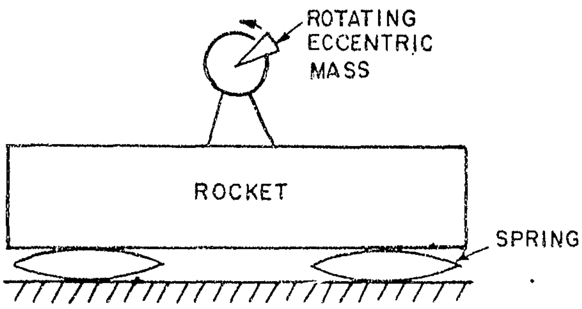


Fig. 2a. Vibrating System.

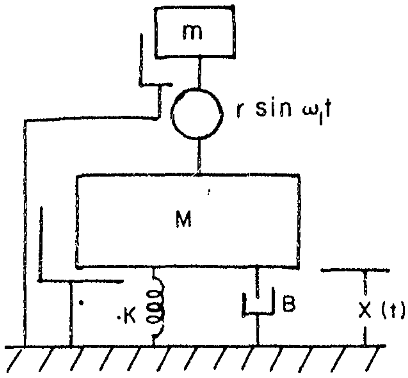


Fig. 2b. Schematic Diagram of Vibrating System.

r = Distance from center of rotation to center of gravity of the eccentric mass

x = Displacement from rest position

ω_1 = Frequency of rotation of eccentric mass

K = Spring constant

B = Damping factor (inherent in a practical system)

M = Mass of rocket motor and rotating motor

m = Mass of eccentric

The system is constrained to motion in the vertical direction.

The equation of motion of the system is

$$(1) \qquad M\ddot{x}(t) + m\left[\ddot{x}(t) + \frac{d^2}{dt^2} r \sin \omega_1 t\right] + B\dot{x}(t) + Kx(t) = 0$$

$$(2) \qquad (M + m)\ddot{x} - mr\, \omega_1^{\,2} \sin \omega_1 t + B\dot{x} + Kx = 0 .$$

The Laplace transform of (2) is

$$(3) \qquad (M + m)\, X(s)\, s^2 + BX(s)\, s + KX(s) = \frac{\dfrac{mr\, \omega_1^{\,2}}{\omega_1}}{1 + \dfrac{s^2}{\omega_1^{\,2}}}$$

and

$$(4) \qquad X(s) = \frac{mr\, \omega_1 \frac{1}{K}}{\left(1 + \dfrac{s^2}{\omega_1^{\,2}}\right)\left(\dfrac{M + m}{K}\, s^2 + \dfrac{B}{K}\, s + 1\right)}$$

Let $\omega = \sqrt{\dfrac{K}{M + m}}$, and $\dfrac{2\,\xi}{\omega} = \dfrac{B}{K}$,

then for the steady state condition, that is, as $t \to \infty$

$$(5) \quad x(t) = \frac{\frac{mr}{K}\, \omega_1^{\,2}\, \omega^2}{\left[(\omega^2 - \omega_1^{\,2})^2 + 4\, \xi^2\, \omega_1^{\,2}\, \omega^2\right]^{1/2}} \sin\, (\omega t - \psi)$$

where

$$(6) \qquad \psi = \tan^{-1} \frac{2\, \xi\, \omega_1\, \omega}{\omega^2 - \omega_1^{\,2}} \quad .$$

When $\omega_1 = \omega$ the system is being driven at the undamped resonant or natural frequency.

Let $\omega_1 = \omega$ and $t \to \infty$.

Then

$$(7) \qquad x(t) = \frac{mr\, \omega^2}{2\, \xi\, K} \sin\, (\omega t - \frac{\pi}{2}) \quad .$$

Equation (7) demonstrates that if the frequency of rotation of the eccentric mass is the natural frequency of the mass-spring system, the phase of the sinusoidal motion of the rocket lags that of the eccentric by 90°, and is independent of the damping factor.

Therefore, if this 90° phase shift is maintained throughout the test period, the system is oscillating at its natural frequency and the mass of the rocket motor can be determined by the equation

$$M = \frac{K}{\omega^2} - m \quad . \tag{8}$$

The problem to be solved is the design of a control system to drive the eccentric mass at an angular rate which corresponds to the resonant frequency of the spring-mass system.

SHAKER SYSTEM

ROCKET MOTOR CHARACTERISTICS

The method described in this paper for measuring the mass of a rocket motor is applicable to rocket motors in general; however, each system must be designed to fit the characteristics of the particular rocket motor.

The characteristics of the rocket motor considered in this system are:

1. Weight of motor before ignition is 25,500 lb.
2. Weight of motor completely burnt is 9,650 lb.
3. Time required to completely expend the propellant is approximately 60 seconds.
4. Motor is equipped with jetavators that deflect the thrust, thereby producing thrust components normal to the main thrust vector of approximately 10,000 lb. The jetavators are capable of one-cps operation.

DETERMINATION OF SUSPENSION-SYSTEM CONSTANTS

The fastening of the front of a rocket motor to a push rod in a static firing test stand essentially defines a mass-spring system. The designers of such stands try to make the natural frequency as high as possible. Because of flexure in the motor cases and in the push rods in such test stands, there exists a limit in the maximum natural frequency that can be obtained. It has been determined empirically that for a 25,000-lb motor the natural frequency is between 20 and 25 cps.

Therefore the natural frequency of the system in the vertical direction should be sufficiently below 20 cps so that the motion caused by forces which can couple into the system at those frequencies will be attenuated and have negligible effect. It is also desirable to have a high natural frequency in the vertical system, in order that as much frequency information as possible can be extracted from the system. A natural frequency for the unburnt motor of 10 cps was chosen as a compromise between these two considerations.

SPRING CONSTANT

From the equation $\omega = \sqrt{\frac{K}{M}}$ we get

(9) $$K = \frac{\omega^2 W}{g}$$

where

ω = Frequency in rad/sec

K = Spring constant in lb/ft

$M = \frac{W}{g}$ = Mass of the system in slugs.

Then

(10) $$K = \frac{(2\pi)^2 (10)^2 (25{,}500)}{32.2} = 3.125 \times 10^6 \text{ lb/ft}$$

NATURAL FREQUENCY RANGE

The natural frequency of the system when the rocket motor is unburnt is

(11) $$\omega = 2\pi\ (10) = 62.8 \text{ rad/sec}$$,

and for the burnt condition

$$\omega = \sqrt{\frac{Kg}{W}} = \sqrt{\frac{3.125 \times 10^6 \ (32.2)}{9650}} = 102.0 \text{ rad/sec} \quad . \tag{12}$$

The frequencies in cps are then 10.0 and 16.25, respectively.

SYSTEM DAMPING RATIO

To start the design of the system, the damping was estimated to be 15 percent of critical when the rocket motor was unburnt, and 10 percent when the motor was burnt; the difference in damping is attributed to the energy absorbed by the propellant grain. A very low damping ratio is desirable in order that reasonable vertical displacement can be achieved with little expended energy. If these estimates prove excessive in error, major redesign is required.

REQUIRED MASS UNBALANCE

A limit of 1-g peak acceleration was set for the system. Tests on SNORT (Supersonic Naval Ordnance Research Track) had produced accelerations of this magnitude in a series of sled tests involving the rocket motor being considered in this paper; there appeared to be no untoward acceleration effects on the performance of these motors during these tests.

Referring to Equation (7), and differentiating twice, we get the vertical acceleration of the rocket motor when excited at the natural frequency ω,

$$a(t) = \frac{-mr\,\omega^4}{2\,\xi\,K} \sin\left(\omega t - \frac{\pi}{2}\right) \tag{13}$$

hence the peak vertical acceleration is $\frac{mr\,\omega^4}{2\,\xi\,K}$.

The maximum value of (mr) in order that the system not exceed a peak acceleration of 1 g at $\omega = 62.8$ rad/sec is

$$(14) \qquad (mr)_{10\ cps} = \frac{2\zeta\ Kg}{\omega^4} = \frac{2(.15)(3.125)(10^6)(32.2)}{(62.8)^4}$$

$$= 1.87 \text{ slug-ft.}$$

However, an existing design for an eccentric mass produced a maximum mr = .877 slug-ft, and was adopted for use in the mass measuring system.

The maximum value of (mr) in order that the system not exceed a peak acceleration of 1 g at $\omega = 102.0$ rad/sec is

$$(15) \qquad (mr)_{16.25\ cps} = \frac{2(.10)(3.125 \times 10^6)(32.2)}{(102)^4}$$

$$= .186 \text{ slug-ft.}$$

A device to reduce the value of mr in order to keep the maximum acceleration loading to below 1 g has been designed for incorporation in the system.

DETERMINATION OF DAMPING CONSTANT B

The larger damping factor existing in the system before the motor propellent is spent is attributed to deformation of the plastic propellent under vibration, which thus absorbs energy. We therefore assume that the damping changes directly with the mass of the motor. Frequency being inversely proportional to the square root of the mass, we can write

$$(16) \qquad B = K_1 + \frac{K_2}{f^2}$$

where K_1 and K_2 are to be determined. We have the damping at the beginning of burning and at the end of burning. Therefore

(17) $$14920 = K_1 + (.01)K_2 \quad ,$$

(18) $$6120 = K_1 + (.00377)K_2 \quad ,$$

and

(19) $$K_2 = 1.41 \times 10^6 \ \frac{\text{lb}}{\text{ft-sec}}$$

and

(20) $$K_1 = 820 \ \frac{\text{lb-sec}}{\text{ft}} \quad .$$

The damping constant at 12 and 14 cps is then

(21) $$B_{f\,=\,12\text{ cps}} = 820 + \frac{1.41 \times 10^6}{144} = 10620 \ \frac{\text{lb-sec}}{\text{ft}} \quad .$$

(22) $$B_{f\,=\,14\text{ cps}} = 820 + \frac{1.41 \times 10^6}{196} = 8020 \ \frac{\text{lb-sec}}{\text{ft}} \quad .$$

DETERMINATION OF MASS UNBALANCE FOR VARIOUS VALUES OF NATURAL FREQUENCY

The maximum value of mr when $\omega = 14$ cps is

(23) $$(mr) = \frac{32.2\ B}{\omega^3} = \frac{32.2(8020)}{(14 \times 6.28)^3} = \frac{258{,}300}{(87.8)^3} = .382 \text{ slug-ft.}$$

The maximum value of mr when $\omega = 12$ cps is

(24) $$(mr) = \frac{32.2(10620)}{(12\text{x}6.28)^3} = \frac{342{,}000}{(75.4)^3} = .797 \text{ slug-ft.}$$

A plot of the mr required to keep the acceleration below 1 g, and the damping constant B versus the frequency of oscillation, is shown in Fig. 3.

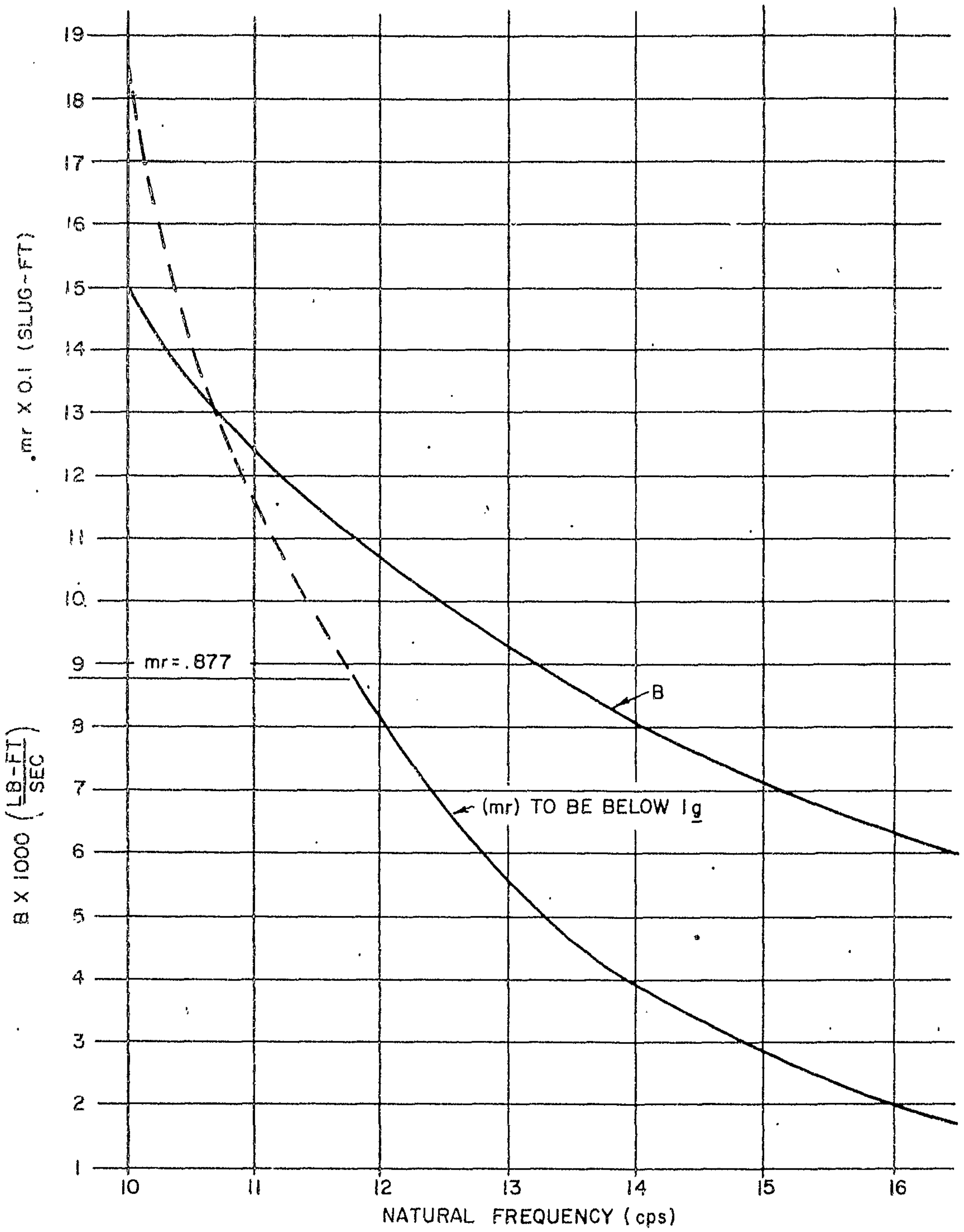


Fig. 3. Damping Factor, B, and Product of Mass and Center of Gravity, as Functions of Natural Frequency, ω.

POWER REQUIRED TO DRIVE SHAKER SYSTEM

The energy required to keep a spring-mass system oscillating is the same as the energy consumed by the damping in the system.

The equation for power is

(25) $$p(t) = f(t)\ v(t) \quad ,$$

but

(26) $$f(t) = B\dot{x}(t) = Bv(t) \qquad B = \text{damping constant.}$$

Substituting, we get

(27) $$p(t) = Bv(t)^2 \quad .$$

The velocity is a sine function, and we may therefore substitute the RMS value of $v(t)$ to obtain the average power

(28) $$P = BV^2$$ V is the RMS value of $v(t)$

P is the average power

The peak value of velocity is obtained by differentiating the position given in Equation (7) with respect to time. Therefore

(29) $$V_p = \frac{mr\ \omega^3}{2\ \xi\ K} = \frac{mr\ \omega^2}{B} \quad .$$

The RMS velocity is then, $\dfrac{mr\ \omega^2}{\sqrt{2}\ B}$; and the average power is

(30) $$P = \left(\frac{mr\ \omega^2}{\sqrt{2}\ B}\right)^2 B = \frac{(mr)^2\ \omega^4}{2B} \quad .$$

The power required by the shaker system, plotted against the resonant frequencies of the system, is shown in Fig. 4. The values of mr and B corresponding to the resonant frequencies are taken from the curves of Fig. 3.

Figure 4 shows that the maximum power required by the system occurs when mr = .877 slug-ft and the peak acceleration is 1 g. The resonant frequency at maximum power is 11.8 cps. The power is 1050 ft-lb/sec. The horsepower is given by:

$$\text{horsepower} = \frac{1050}{550} = 1.92 \quad . \tag{31}$$

A 5 hp motor was chosen to drive the system. The extra (more than double) power is necessary to take care of losses in the drive system and to bring the system up to speed in a short period of time.

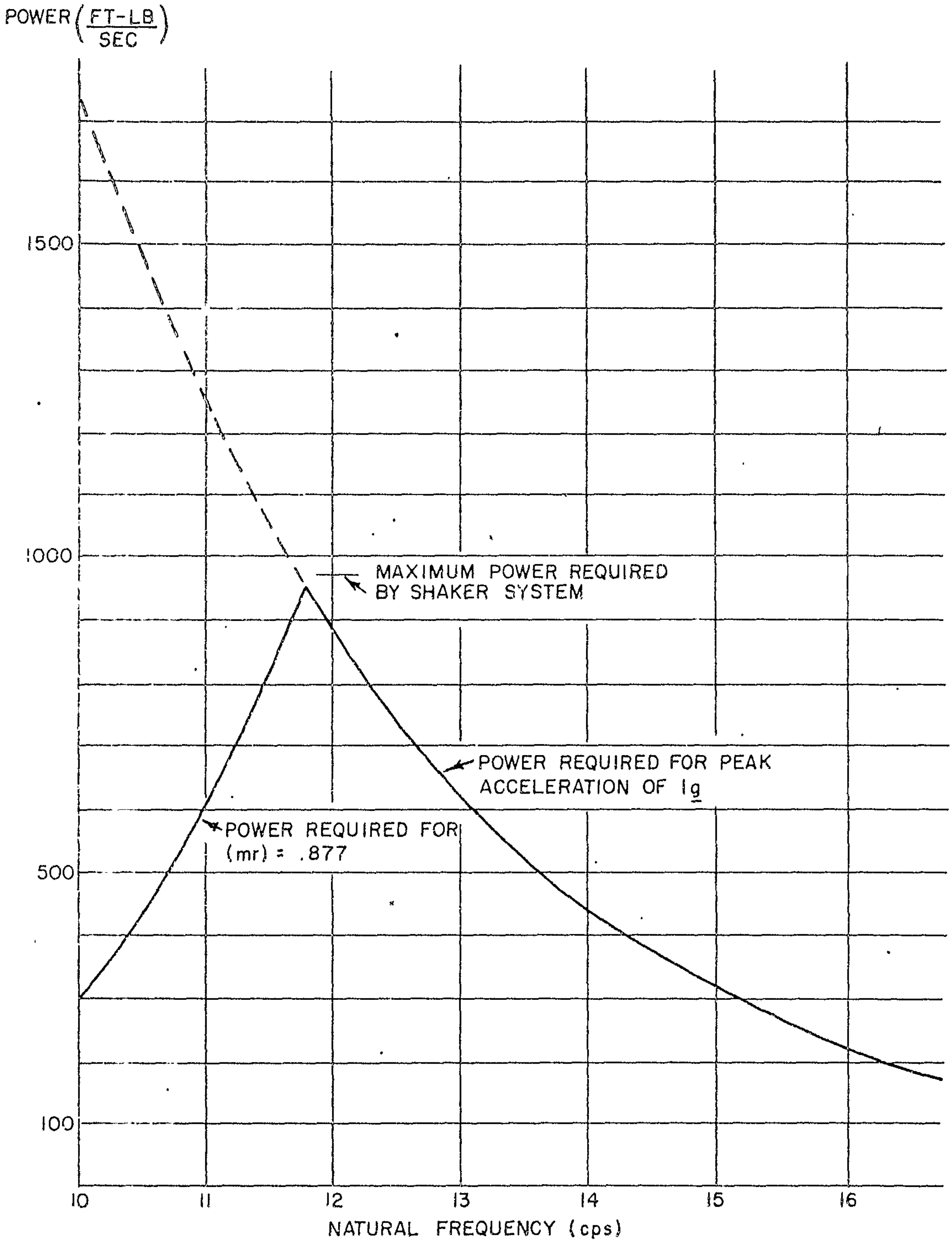


Fig. 4. Power Required by Shaker System, Versus Natural Frequency

CONTROL SYSTEM REQUIREMENTS

STEADY STATE ACCURACY

A question that may be asked is, Why not measure the phase between the driving function and the oscillation of the mass-spring system to determine the natural frequency of the system, and not bother driving the system at its natural frequency? The answer to this is that in order to determine the natural frequency from phase information, the spring constant and damping factor must be known. The spring constant can be accurately determined and is constant throughout the burning of the motor. However, the damping of the system is not precisely known during the burning period, and it is only at the natural frequency that this value is not important. Also, it requires a great deal more power to drive a system at a sufficient amplitude when the driving frequency is not near resonance. Therefore, it is necessary that the phase difference be kept as close to 90° as the specified accuracy of the system requires.

The horizontal accelerometer and its associated recording equipment will be no more accurate than 1.5 percent. In considering the evaluation of the equation, $T = Ky + M(t)\ddot{y}$, the value of $M(t)$ should be consistent with this accuracy, and therefore $M(t)$ is to be determined to an accuracy of approximately 1 percent. Referring to Equation (6)

$$\psi = \tan^{-1} \frac{2 \xi \omega_1 \omega}{\omega^2 - \omega_1^2}$$

where ω_1 is the driving frequency,

ω is the natural frequency of the system,

ξ is the damping ratio,

ψ is the phase lag between the driving function and the spring-mass system.

To maintain the overall accuracy of the mass-measuring system to 1 percent, we will attempt to keep the phase lag ψ to a value less than that which would produce an error of 0.5 percent.

Now consider an ω_1

(32) $$\omega_1 = \sqrt{\frac{K}{M}}$$

and an ω

(33) $$\omega = \sqrt{\frac{K}{1.005\ M}} = \frac{1}{1.011}\sqrt{\frac{K}{M}}$$

and therefore

(34) $$\omega_1 = 1.011\ \omega\ .$$

Then, substituting in Equation (6),

(35) $$\psi = \tan^{-1}\frac{2\xi\ (1.011)\omega}{(.011)\omega} = \tan^{-1}\frac{2\xi\ (1.011)}{.011}$$

For a damping ratio of .15, in the spring-mass system, ψ = $87^\circ 55'$ and therefore the servo system must follow to within 2°.

If the estimation of damping is in error by a factor of 2, and ξ = .3, then ψ = $88^\circ 57'$ and the servo system must follow to within 1°.

CHARACTERISTICS OF INPUT SIGNAL RESULTING FROM A SUDDEN CHANGE IN MASS, AND REQUIRED SYSTEM RESPONSE

It is anticipated that certain failures in the solid propellant system, such as large fragments of unburnt propellent being blown through the nozzles, would cause sudden changes in the mass of the system. We then have a condition of a step mass change.

Under normal burning conditions, it is assumed that the changes in frequency of the vibrating system are slow, and that transient effects can be neglected. However, this is not necessarily true in the case of the step change. It is necessary to detect these malfunctions when they occur, and to know what effects such changes have on the mass-measuring system as a whole.

An expression for the motion of the vibrating system when mass is suddenly removed can be found in the following way:

First, assume the system is being driven at resonance, and that all transients have decayed to a negligible value. At any time, $t = a$, evaluate the system and determine the position and velocity of the mass.

Then consider a system that has a different value for the mass, and is driven by the same force function but starting at $t = 0$, and whose initial conditions are the position and velocity of the mass that was found for $t = a$ in the first case.

From Equation (7), the position of a system being driven at resonance by a force, $f(t) = mr\,\omega^2 \sin \omega t$, is

$$x(t) = \frac{mr\,\omega^2}{2\,\xi\,K} \sin\left(\omega t - \frac{\pi}{2}\right) = \frac{mr\,\omega}{B} \sin\left(\omega t - \frac{\pi}{2}\right) .$$

The velocity is found by differentiating the above to give

(36) $$\dot{x}(t) = \frac{mr\,\omega^2}{B} \cos\left(\omega t - \frac{\pi}{2}\right) .$$

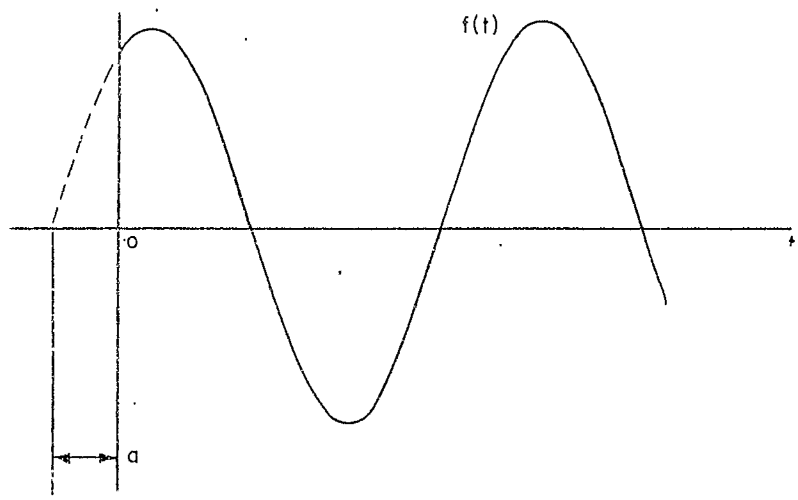
Now if the function is evaluated at $t = a$,

(37) $$x(a) = \frac{mr\,\omega}{B} \sin\left(\omega a - \frac{\pi}{2}\right) = -\frac{mr\,\omega}{B} \cos \omega a ,$$

and

(38) $$\dot{x}(a) = \frac{mr\,\omega^2}{B} \cos\left(\omega a - \frac{\pi}{2}\right) = \frac{mr\,\omega^2}{B} \sin \omega a .$$

Consider now a sine wave advanced in time by "a" second, and starting at zero, as indicated below:



$$f(t) = mr\,\omega^2 \sin\left[\omega(t + a)\right]\mu(t)$$

Fig. 5. Sine Wave Advanced in Time by "a" Second and Starting at Zero Time.

The Laplace transform of $f(t) = mr\,\omega^2 \left[\omega(t + a)\right]\mu(t)$, as indicated by Aseltine[1] is

(39) $$F(s) = mr\,\omega^2 e^{as} \mathcal{L} \sin \omega t(t - a) \quad .$$

This transform, evaluated in Appendix A, is

(40) $$F(s) = \frac{mr\,\omega^3 \cos \omega a\left[\left(\frac{\tan \omega a}{\omega}\right) s + 1\right]}{s^2 + \omega^2} \quad .$$

We can now consider the problem of the force f(t) applied to the system with a new mass, M, and having the initial conditions

(41) $$x(0) = \frac{-mr\,\omega}{B} \cos \omega a$$

(42) $$\dot{x}\,(0) = + \frac{mr\,\omega^2}{B} \sin \omega a \quad .$$

Then

(43) $$F(s) = M\,X(s)s^2 - M\,X(0)s - M\,\dot{X}(0) + B\,X(s)s - B\,X(0) + K\,X(s) \quad .$$

Substituting for F(s),

(44) $$\frac{mr\,\omega \cos \omega a\left[\left(\frac{\tan \omega a}{\omega}\right) s + 1\right]}{1 + \frac{s^2}{\omega^2}} = M\,X(s)s^2 + \frac{M\,mr\,\omega}{B} \cos\,(\omega a)s - \frac{M\,mr\,\omega^2}{B} \sin \omega a + B\,X(s)s + mr\,\omega \cos \omega a + K\,X(s)$$

and

(45) $$X(s) = \frac{\dfrac{mr\omega \cos \omega a\left[\left(\frac{\tan \omega a}{\omega}\right) s + 1\right]}{\left(1 + \frac{s^2}{\omega^2}\right)} - \dfrac{Mmr\omega}{B} \cos\,(\omega a)s}{M\,s^2 + B\,s + K} + \frac{\dfrac{Mmr\,\omega^2}{B} \sin \omega a - mr\omega \cos \omega a}{M\,s^2 + B\,s + K}$$

$$= \frac{mr\omega \cos \omega a}{K}\left[\frac{\left(\frac{\tan \omega a}{\omega} s + 1\right)}{\left(\frac{1 + s^2}{\omega^2}\right)\left(\frac{M}{K} s^2 + \frac{B}{K} s + 1\right)} - \frac{\frac{M}{B} s - \frac{M\omega}{B} \tan \omega a + 1}{\left(\frac{M}{K} s^2 + \frac{B}{K} s + 1\right)}\right] .$$

If $a = \theta$, then

$$(46) \quad x(s) = \frac{mr\omega}{K}\left[\frac{1}{\left(\frac{M}{K} s^2 + \frac{B}{K} s + 1\right)\left(1 + \frac{s^2}{\omega^2}\right)} - \frac{\frac{M}{B} s + 1}{\left(\frac{M}{K} s^2 + \frac{B}{K} s + 1\right)}\right] .$$

The time function as given by Nixon[2] is

$$(47) \quad (t) = \frac{mr\omega}{K}\left\{\frac{\omega^2 \omega_2^2}{\left[(\omega^2 - \omega_2^2)^2 + 4 \xi^2 \omega^2 \omega_2^2\right]^{1/2}}\left[\frac{1}{\omega} \sin (\omega t - \psi_1)\right.\right.$$

$$\left. + \frac{1}{\omega_2\sqrt{1 - \xi^2}} e^{-\xi\omega_2 t} \sin \left(\omega_2\sqrt{1 - \xi^2}\, t - \psi_2\right)\right]$$

$$\left. - \omega_2\left[\frac{1 - 2 \frac{M}{B} \xi \omega_2 + \frac{M^2}{B^2} \omega_2^2}{1 - \xi^2}\right]^{1/2} e^{-\xi \omega_2 t} \sin \left(\omega_2\sqrt{1 - \xi^2}\, t + \psi_3\right)\right\}$$

where

$$\psi_1 = \tan^{-1} \frac{2 \xi \omega \omega_2}{\omega_2^2 - \omega^2} , \qquad \psi_2 = \tan^{-1} \frac{-2\xi \omega_2^2\sqrt{1 - \xi^2}}{\omega^2 - \omega_2^2(1 - 2 \xi^2)} ,$$

$$\psi_3 = \tan^{-1} \frac{\frac{M}{B} \omega_2\sqrt{1 - \xi^2}}{1 - \frac{M}{B} \xi \omega_2} \qquad \text{and,} \qquad \omega_2 = \sqrt{\frac{K}{M}} .$$

If ω_2 is set 10 percent higher than ω (that is, $1.1\omega = \omega_2$) and ξ is set at .1, then

$$(48)\quad x(t) \approx \left[\frac{mr\omega}{K}\, 4.0\omega \sin(\omega t - 49^\circ) + 3.64\omega\, e^{-\xi \omega_2 t} \sin(\omega_2 t - 232^\circ) - 5.0\omega\, e^{-\xi \omega_2 t} \sin(\omega_2 t + 84.3^\circ)\right] .$$

The exponential terms, when added vectorially as indicated in Fig. 6, give a single exponential term; the expression for x(t) is then

$$(49)\quad x(t) \approx \frac{mr\omega^2}{K}\left[4.0 \sin(\omega t - 49^\circ) + 3.5\, e^{-\xi \omega_2 t} \sin(\omega_2 t + 218^\circ)\right].$$

A plot of the two terms is shown in Fig. 7.

If $a = \frac{\pi}{2\omega}$, then

$$(50)\quad x(t) = \frac{mr\omega}{K}\left\{\frac{\omega^2 \omega_2^2}{\left[(\omega^2 - \omega_2^2)^2 + 4\xi^2\omega^2\omega_2^2\right]^{1/2}}\left[\frac{1}{\omega}\cos(\omega t - \psi_1) + \frac{1}{\omega\sqrt{1-\xi^2}}\, e^{-\xi\omega_2 t}\sin\left(\omega_2\sqrt{1-\xi^2}\,t - \psi_2\right)\right] + \frac{\omega_2 \frac{M\omega}{B}}{\sqrt{1-\xi^2}}\, e^{-\xi\omega_2 t}\sin \omega_2\sqrt{1-\xi^2}\,t\right\}$$

where

$$\psi_1 = \tan^{-1}\frac{2\xi\omega\omega_2}{\omega_2^2 - \omega^2}$$

$$\psi_2 = -\tan^{-1}\frac{1-\xi^2}{-\xi} - \tan^{-1}\frac{2\xi\omega_2^2\sqrt{1-\xi^2}}{\omega^2 + 2\xi^2\omega_2^2 - \omega_2^2} .$$

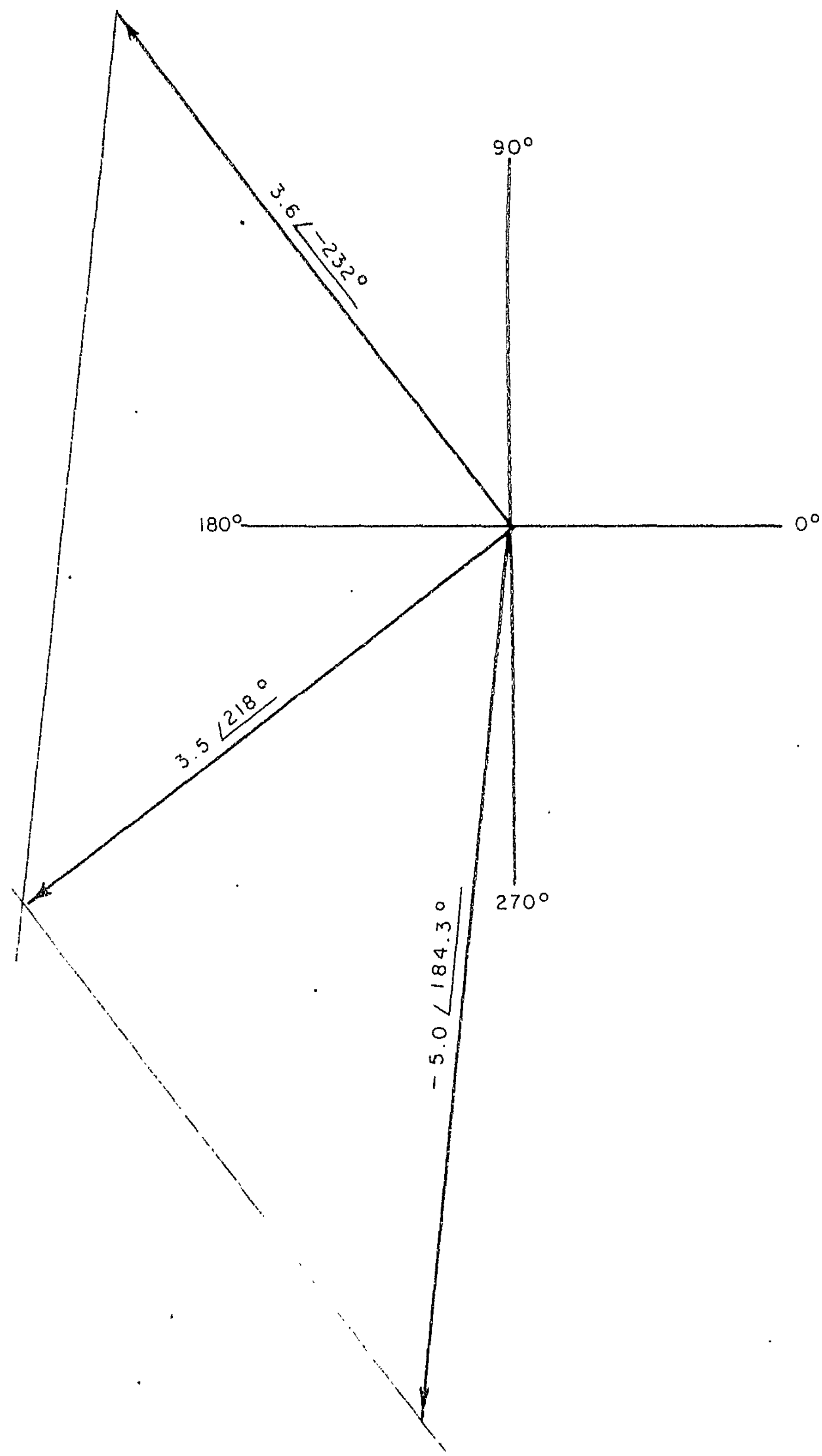


Fig. 6. Vector Addition of Transient Terms When the Mass Change Occurs at the Positive Going Zero Value of the Force Function.

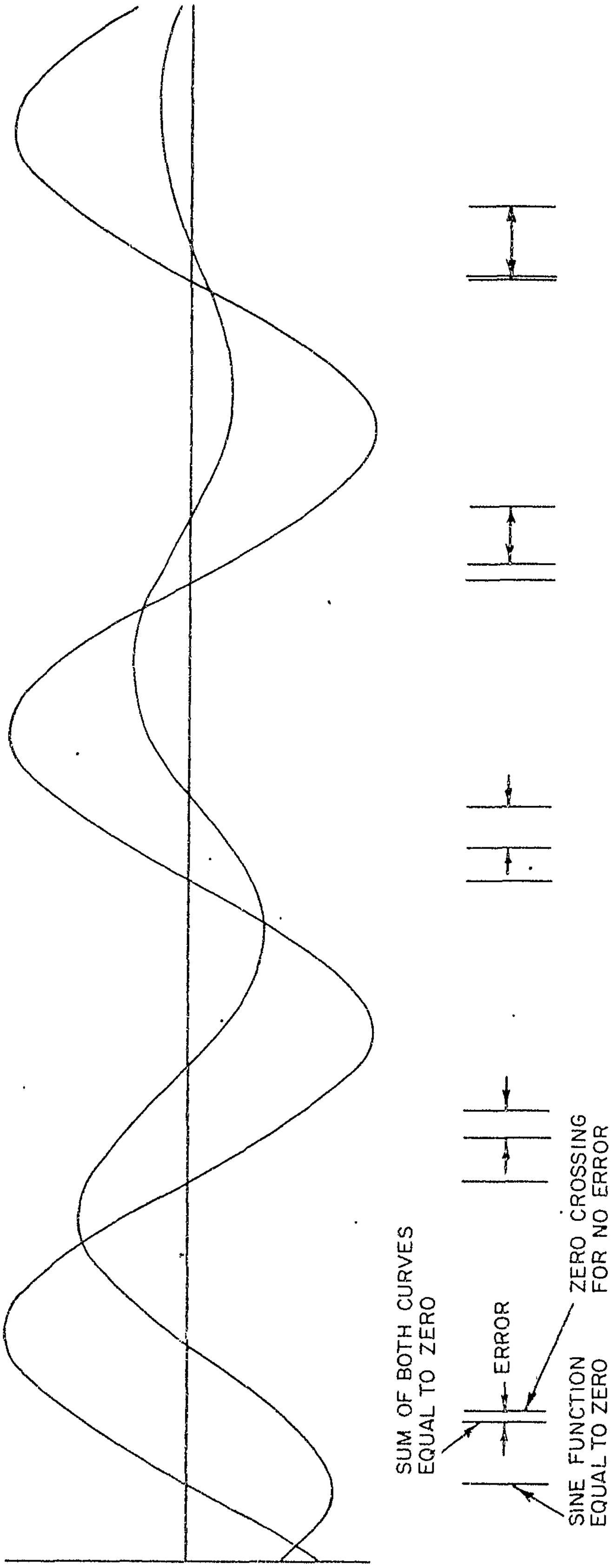


Fig. 7. Transient and Steady-State Terms of the Oscillating Mass-Spring System When a Step Change in Mass Occurs at a Positive Going Zero Value of the Force Function.

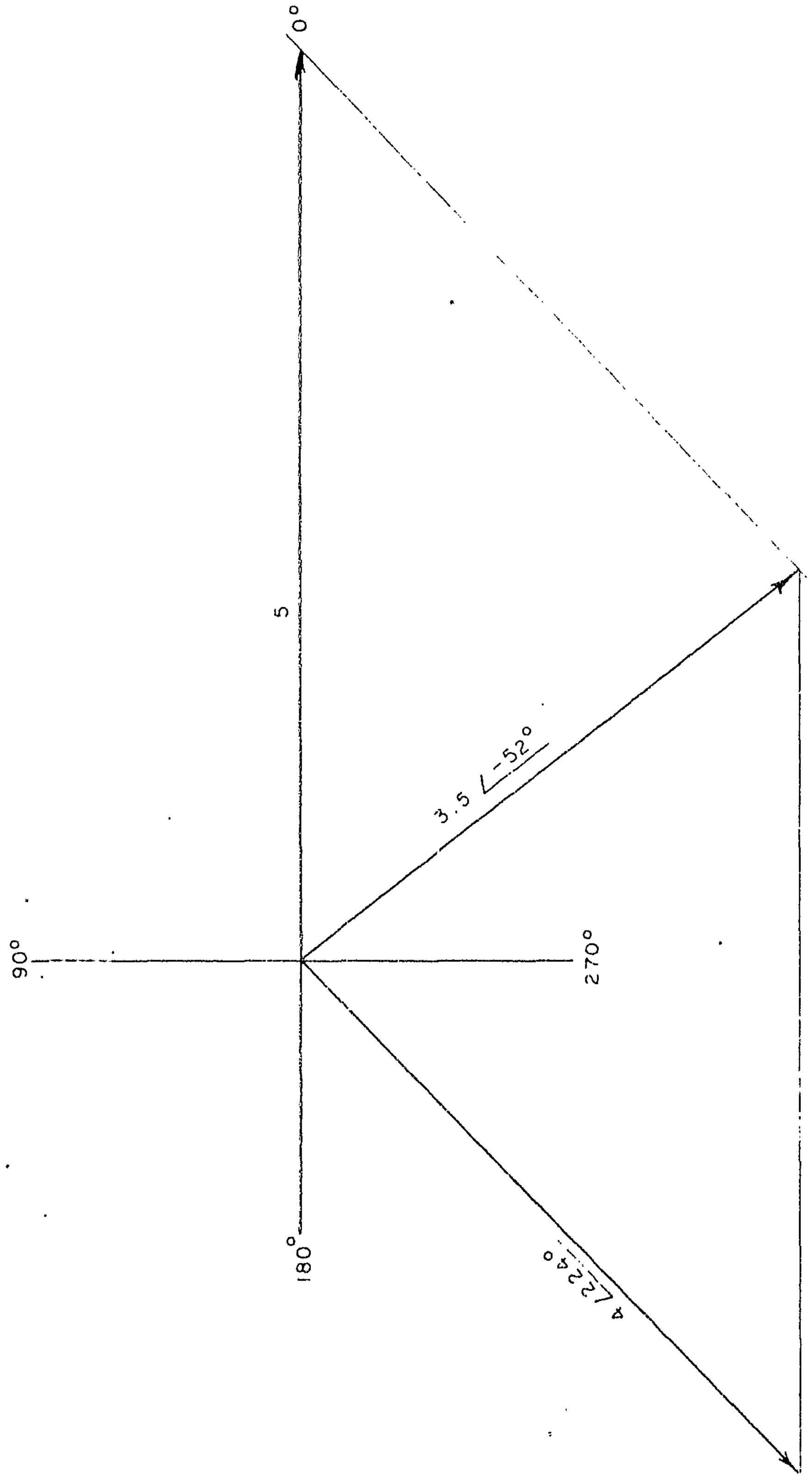


Fig. 8. Vector Addition of Transient Terms When Mass Change Occurs at Peak Value of the Force Function.

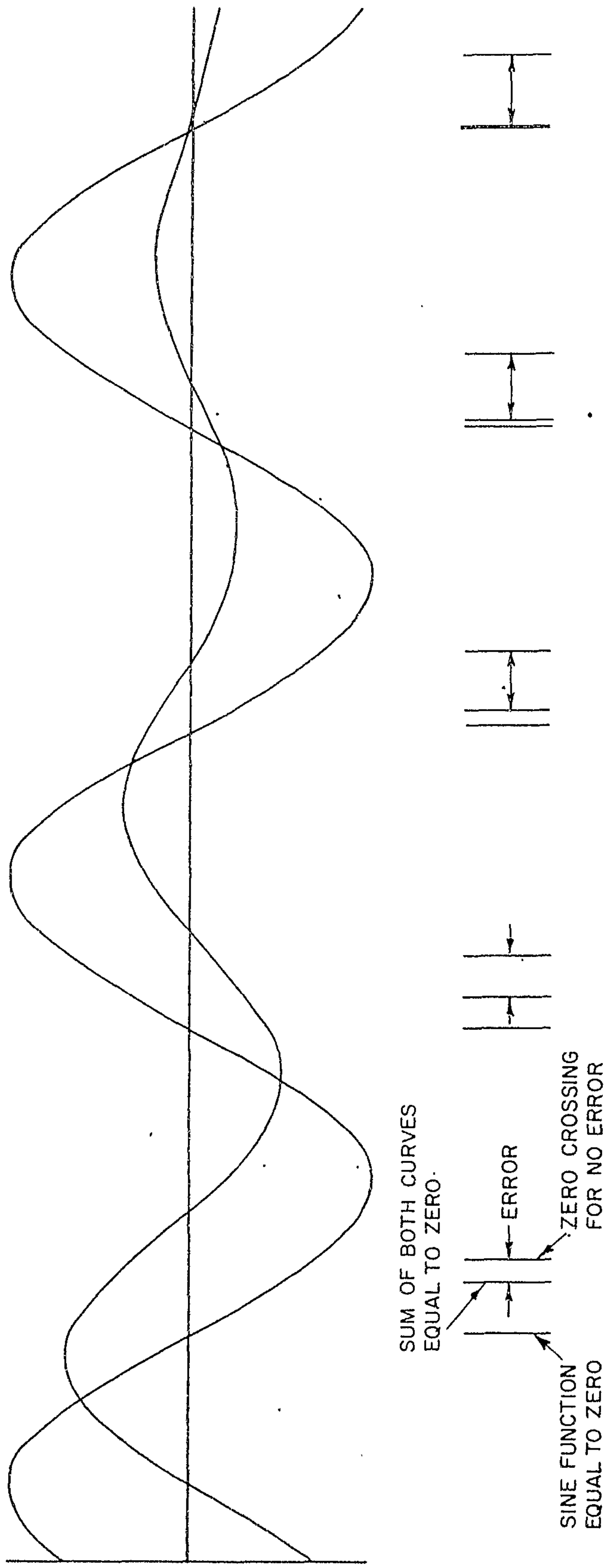


Fig. 9. Transient and Steady-State Terms of Oscillating Mass-Spring System When Step Change in Mass Occurs at Peak Value of Force Function.

The time function, given the same assumption as in the a = 0 case, is

$$(51) \quad x(t) \approx \frac{mr\omega^2}{K}\left[4 \cos (\omega t - 49^\circ) + 4\, e^{-\xi \omega_2 t} \sin (\omega_2 t + 224^\circ) + 5\, e^{-\xi \omega_2 t} \sin (\omega_2 t)\right] .$$

Adding the exponental term vectorially in Fig. 8, the resulting expression for x(t) is

$$(52) \quad x(t) \approx \frac{mr\omega^2}{K}\left[4 \cos (\omega t - 49^\circ) + 3.5\, e^{-\xi \omega_2 t} \sin (\omega_2 t - 52^\circ)\right] .$$

A plot of the two terms is shown in Fig. 9.

An examination of Figs. 7 and 9 show that if phase measurements are made when the time functions are zero, the phase shift approaches its maximum value in what appears to be an exponential manner, described by the equation

$$(53) \qquad \phi = 1 - e^{-\frac{t}{\xi \omega_2}}$$

where ϕ is the phase shift and $\xi \omega_2$ is the time constant.

To verify the results of the analysis of the system for a sudden mass decrease, an electrical analogue was constructed and a rapid change in inductance was used to simulate the mass change.

Given the following electrical system (Fig. 10),

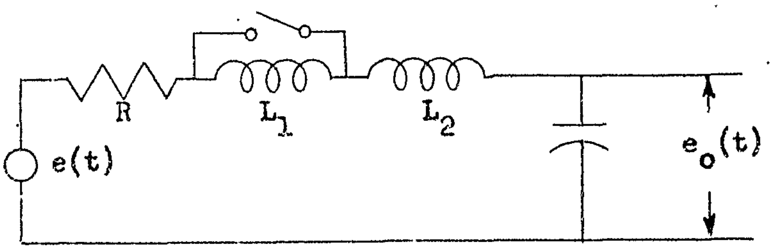


Fig. 10. Analogue System.

the Laplace equation for the output voltage, $e_o(t)$, is

$$(54) \qquad e_o(s) = \frac{E(s)}{s^2 LC + sRC + 1} \qquad \text{where } L_1 + L_2 = L \ .$$

The Laplace equation for the displacement of the mass in a mass-spring-damping system is

$$(55) \qquad X(s) = \frac{\frac{F(s)}{K}}{s^2 \frac{M}{K} + \frac{sB}{K} + 1} \ .$$

Therefore the output voltage of the electrical circuit is analogous to the displacement of the mechanical circuit if we consider the inductance analogous to the mass, the reciprocal of capacitance analogous to the spring constant, and resistance analogous to the damping constant. The values of inductance and capacitance were chosen to produce a resonance in the neighborhood of 10 cps because the recording equipment was suited to that frequency. Damping of about .1 to .15 percent critical was required to simulate the mechanical system. With components that were available, the following system (Fig. 11) was constructed.

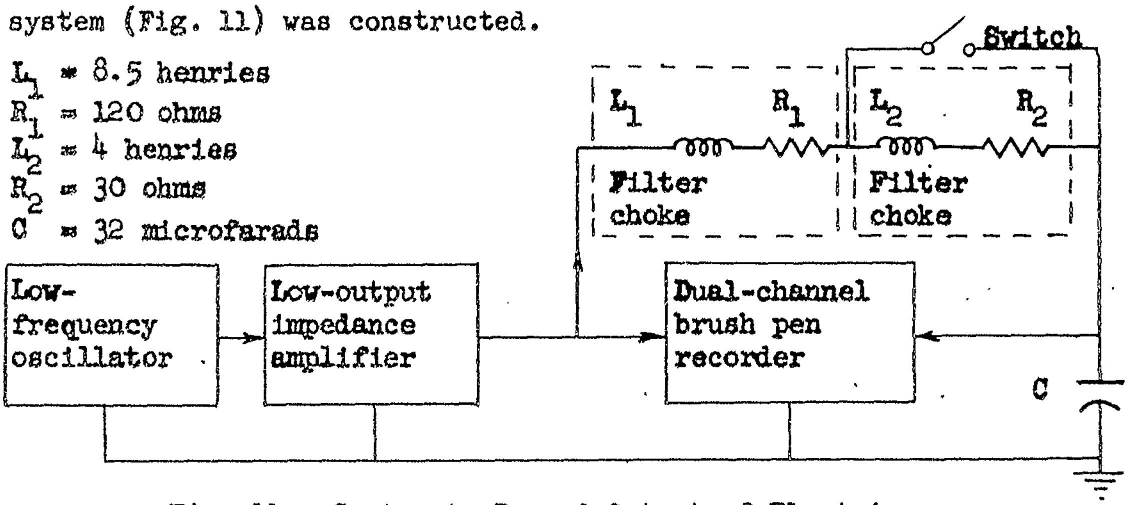


Fig. 11. System to Record Output of Electric Analogue for Step Change of Inductance.

The undamped resonant frequency with both inductors in the circuit is

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{6.28\sqrt{(12.5)(32)(10^{-6})}} = 8 \text{ cps} \quad . \tag{56}$$

The undamped resonant frequency with the 8.5-henry inductor alone is

$$f = \frac{1}{6.28\sqrt{(8.5)(32)(10^{6})}} = 10 \text{ cps} \quad . \tag{57}$$

The damping ratio ξ with 12.5-henry inductance and 150 Ω resistance is given by

$$RC = \frac{2\xi}{\omega} \tag{58}$$

$$\xi = \frac{RC\omega}{2} = \frac{(150)(32)(10^{-6})(50)}{2} = .12 \quad . \tag{59}$$

The damping ratio ξ with 8.5-henry inductance and 120 Ω resistance is

$$\xi = \frac{(120)(32)(10^{-6})(61)}{2} = .12 \quad . \tag{60}$$

The oscillator frequency was adjusted so that the output voltage $e_o(t)$ lagged $e(t)$ by 90° when both L_1 and L_2 were in the circuit. L_1 was then removed and put back into the circuit many times at random, and the input and output voltages were recorded. Each time L_1 was added or subtracted, the phase shifted in the exponential manner indicated by the analysis and with a time constant that agreed with the value $\frac{1}{\xi\,\omega_2}$.

REQUIRED SYSTEM RESPONSE

An overshoot in the response of the control system will indicate when a step mass change occurs. But it is also desired to have the error settle to the steady-state value as quickly as possible. A nominal overshoot of 10 percent is generally chosen for a system in order to have a low settling time for a step input.

It is therefore desired that the control system have a similar overshoot when the exponential transient produced by a step mass change occurs.

UNWANTED SIGNAL DISCRIMINATION

There are several sources of signals that are likely to be produced by the rocket motor and spring-mass system.

As previously described, the resonant frequency of the test stand in the thrust direction is between 20 and 25 cps. Any forces at these frequencies that couple into the vertical system produce motion that has an amplitude of at least 10 db below the natural frequency motion for the same value of force amplitude. It is assumed that coupling from the horizontal to the vertical will be small, and that therefore the effects of these high frequencies can be neglected. If this assumption is not valid, then higher frequencies could not be neglected because the signals to the control system are to be sampled, and aliasing error could arise even though the control system has a low, high-frequency cut-off.

However, unwanted low-frequency signals must be attenuated. These signals arise from the fact that the motor thrust is not

exactly perpendicular to the vertical, and that in fact the thrust axis is likely to wander in small angles about the assumed axis. Also, the motor with which we are concerned in this paper has thrust deflectors (jetavators) that produce a 10,000-lb side thrust. The cycling rate of the deflector can be as high as 1 cps.

From the equation $\omega_1 = 1.011\ \omega$, for a mass measurement of 0.5 percent accuracy, the frequency must be determined to within 1 percent.

The period, T, of a sine wave is

$$T = \frac{2\pi}{\omega} \tag{61}$$

and

$$dT = \frac{-2\pi}{\omega^2}\, d\omega \quad . \tag{62}$$

For small differences the equation becomes

$$\Delta T = \frac{-2\pi}{\omega^2}\, \Delta\omega \quad . \tag{63}$$

For $\Delta\omega = .01\ \omega$, and $\omega = 102$ rad/sec, ΔT becomes

$$\Delta T = \frac{-2\pi(.01)}{102} = 630\ \mu\text{sec} \quad . \tag{64}$$

Consider the system when the resonant frequency is 102 rad/sec. The force supplied by the eccentric mass is then

$$f(t) = mr\omega^2 \sin \omega t = (.187)(102)^2 \sin 102t = 1950 \sin 102t \quad . \tag{65}$$

Let the voltage amplitude, E, derived from the transducer detecting the amplitude of the motor's vertical motion, be one volt.

Now consider the force, $f_T(t) = 10{,}000 \sin 6.28t$, acting on the same system. From the frequency-response curves of a second-order system it is seen that the response at 1 cps is approximately 13 db below that at resonance if the damping ratio is 0.1. Therefore the voltage amplitude derived from the transducer with this force applied would be

$$E_T = \frac{.224 \times 10{,}000}{1950} = 1.15 \text{ volts} \tag{66}$$

where .224 = -13 db.

The slope of the 102 rad/sec voltage at the positive zero crossover is

$$\left.\frac{dE}{dt}\right|_{t\,=\,0} = 102 \frac{\text{volts}}{\text{sec}} \quad . \tag{67}$$

Considering this crossover point, $\Delta E = 102\ \Delta t$.

To keep the period accuracy to within 630 μsec, the zero crossover points of the 16-cps signal must be held to half the 630 μsec.

Then

$$\Delta E = 102 \frac{\Delta t}{2} \text{ volts} = \frac{(102)(630)10^{-6}}{2} \text{ volts} = .032 \text{ volts} \quad . \tag{68}$$

Therefore, the 1-cps signal must be reduced by the ratio $\frac{.032}{1.15}$ = .028 or by -31 db. The input to the control system will have to be filtered in some manner to accomplish this frequency rejection.

CONTROL SYSTEM

POSITION SERVOMECHANISM

Consider the system whereby $\dot{\theta}_o$ is the output of a linear position servomechanism driving the unbalanced rotor. Also, consider the sinusoidal function of the suspended system as emanating from a rotation that has a position θ_{in}. Under steady-state conditions the frequencies of rotation of the two systems are the same. Therefore, the phase difference between sin θ_o and sin θ_{in} can be considered as a position error, $\theta_o - \theta_{in}$.

SAMPLED DATA SYSTEM

Since the amplitude of the vibrating system is not constant, the phase of the two sinusoidal functions can at best be compared every time the sine waves cross zero, or twice every cycle. Comparing the phase twice each cycle of oscillation is equivalent to sampling the position error. The sampled-data position servomechanism is shown in Fig. 12.

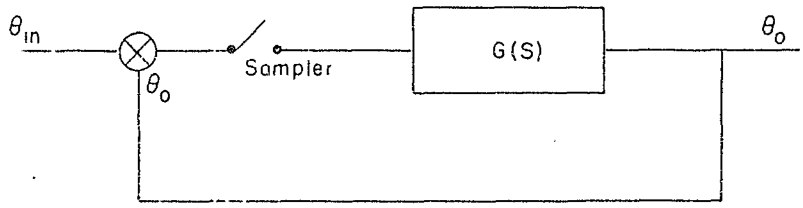


Fig. 12. Sampled-Data Position Servomechanism.

An armature-controlled dc motor is chosen to drive the eccentric mass; the transfer function of the motor[3] is given by

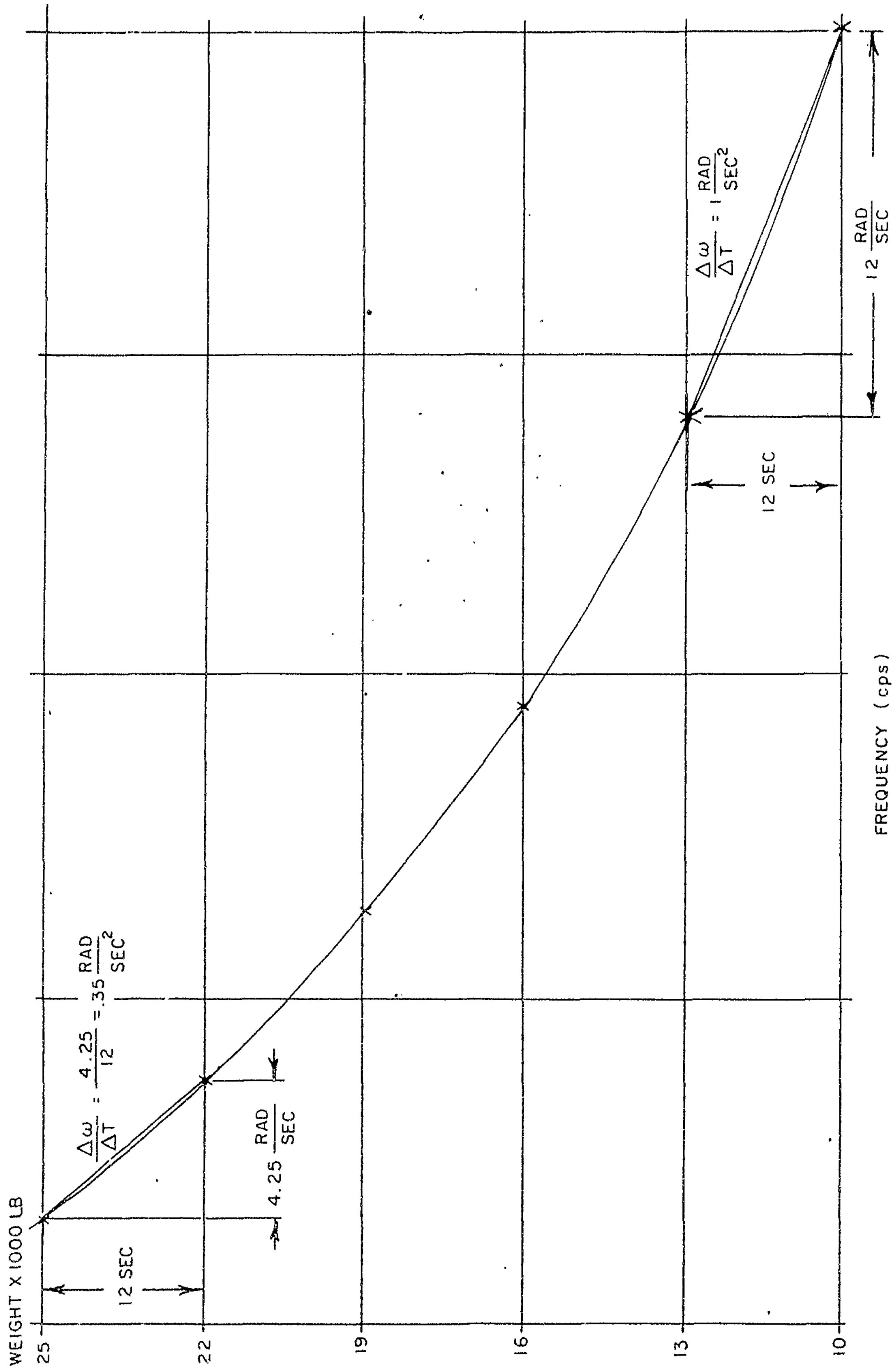


Fig. 13. Natural Frequency of Mass-Spring System as a Function of Mass.

$$G(s)_m = \frac{K_1}{s(as + 1)} \quad (69)$$

The amplifier to drive the 5-hp dc motor and inertial load is a 60-cps magnetic amplifier having a time constant comparable to that of the motor; the amplifier transfer function is given by

$$G(s)_a = \frac{K_2}{(bs + 1)} \quad (70)$$

SECOND-ORDER SYSTEM

A plot of the natural frequency of the system as a function of the mass is shown in Fig. 13. If it is assumed that the mass change is a direct function of time, then for a small increment of time, $\omega(t) = kt$, and by integration, $\theta(t) = \frac{kt^2}{2}$. Therefore, for a non-increasing position error for the indicated input, a double integration is required in the forward loop of the control system. The system with the added integrator, magnetic amplifier, and dc motor is shown below in Fig. 14.

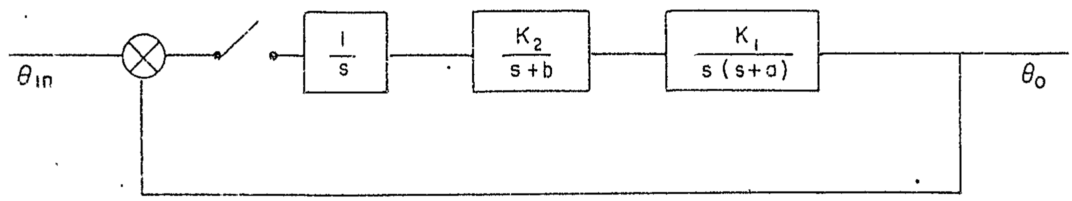


Fig. 14. Uncompensated Control System.

The above system is a second-order servomechanism, and without compensation it is inherently unstable. Since an operational amplifier will be used to provide one of the integrations in the system, it is a simple matter to get a transfer function[4] of the form

$$G(s)_1 = \frac{K_3\,(bs + 1)(cs + 1)}{s} \tag{71}$$

and thereby cancel one of the poles of the amplifier-motor transfer function. The system to be analyzed is shown below in Fig. 15.

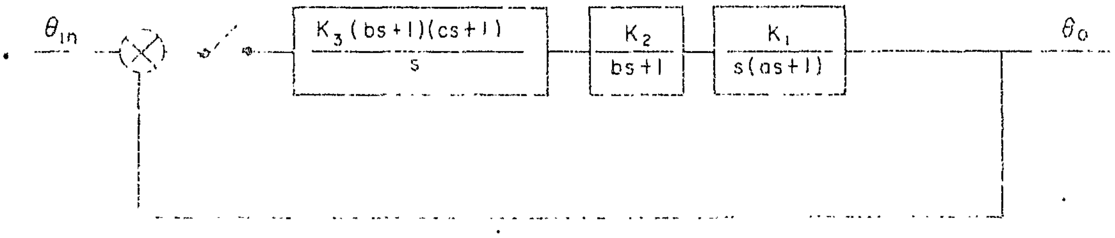


Fig. 15. Compensated Control System.

The value of $G(s)_m G(s)_a$, as given by the manufacturer of the 5-hp motor and driving amplifier, is

$$G(s)_m G(s)_a = \frac{k}{s(.09s + 1)(.1s + 1)} \tag{72}$$

The mechanization of equation (71) is shown on the right edge of Fig. 17. The values selected produce the transfer function

$$G(s)_1 = \frac{-10(.1s + 1)(s + 1)}{s} \; . \tag{73}$$

These values were selected so that the pole at $s = -10$ of $G(s)_a$ would be cancelled.

FILTERS FOR LOW-FREQUENCY SIGNALS

It is necessary to filter the input to the control system in order to provide 31 db rejection at one cps. But any filter will cause phase shift in the 10-16 cps carrier, and the 90° phase relationship between the forcing function of the eccentric mass and the motion of the rocket motor will not be maintained. It is therefore necessary to provide the same phase in the signal obtained from the eccentric mass output. If identical filters are used to filter the input signal and provide phase shift in the signal from the eccentric output, the 90° phase relationship will be maintained.

A double resistance-capacitance high-pass filter was selected for this purpose because it is easy to select components so as to make both filters nearly identical.

The cross-over frequency of the filters was chosen to be 8 cps. This provides an attenuation of approximately 42 db for 1-cps signals.

SAMPLING SYSTEM AND DETECTION OF PHASE BETWEEN Θ_{in} AND Θ_o

Mounted on the shaft that drives the eccentric mass are two sinusoidal potentiometers. One of these is mounted so as to produce a sine wave whose phase corresponds to the phase of the forcing function. The other one is mounted so as to produce a sine wave that lags the forcing function by 90°.

A linear potentiometer is mounted so as to detect the vertical displacement of the rocket motor.

The outputs of these potentiometers are sinusoidal voltages. These electrical signals are fed to squaring circuits; that is, the sine waves are amplified and the top and bottom of the wave is clipped off. The leading and trailing edges of the square waves correspond to the zero voltage value of the sine waves. The square waveforms are shown in Fig. 16.

The solid square wave of Fig. 16c has a phase shift of less than 90°, and indicates that the system is being oscillated below its natural frequency and that therefore the driving function frequency must be increased.

The dashed square wave of Fig. 16c has a phase shift greater than 90°, and indicates that the system is oscillating at a frequency above its natural frequency and that therefore the driving frequency must be reduced.

To obtain the proper control signal every time the sinusoidal motion of the rocket motor crosses the zero position, the following logic is employed:

The signal to increase the speed of the control motor is given by

$$S_{in} = a\,\bar{b}\,c + \bar{a}\,b\,\bar{c} \quad . \tag{74}$$

The signal to decrease the speed of the control motor is given by

$$S_{de} = a\,b\,\bar{c} + \bar{a}\,\bar{b}\,c \quad . \tag{75}$$

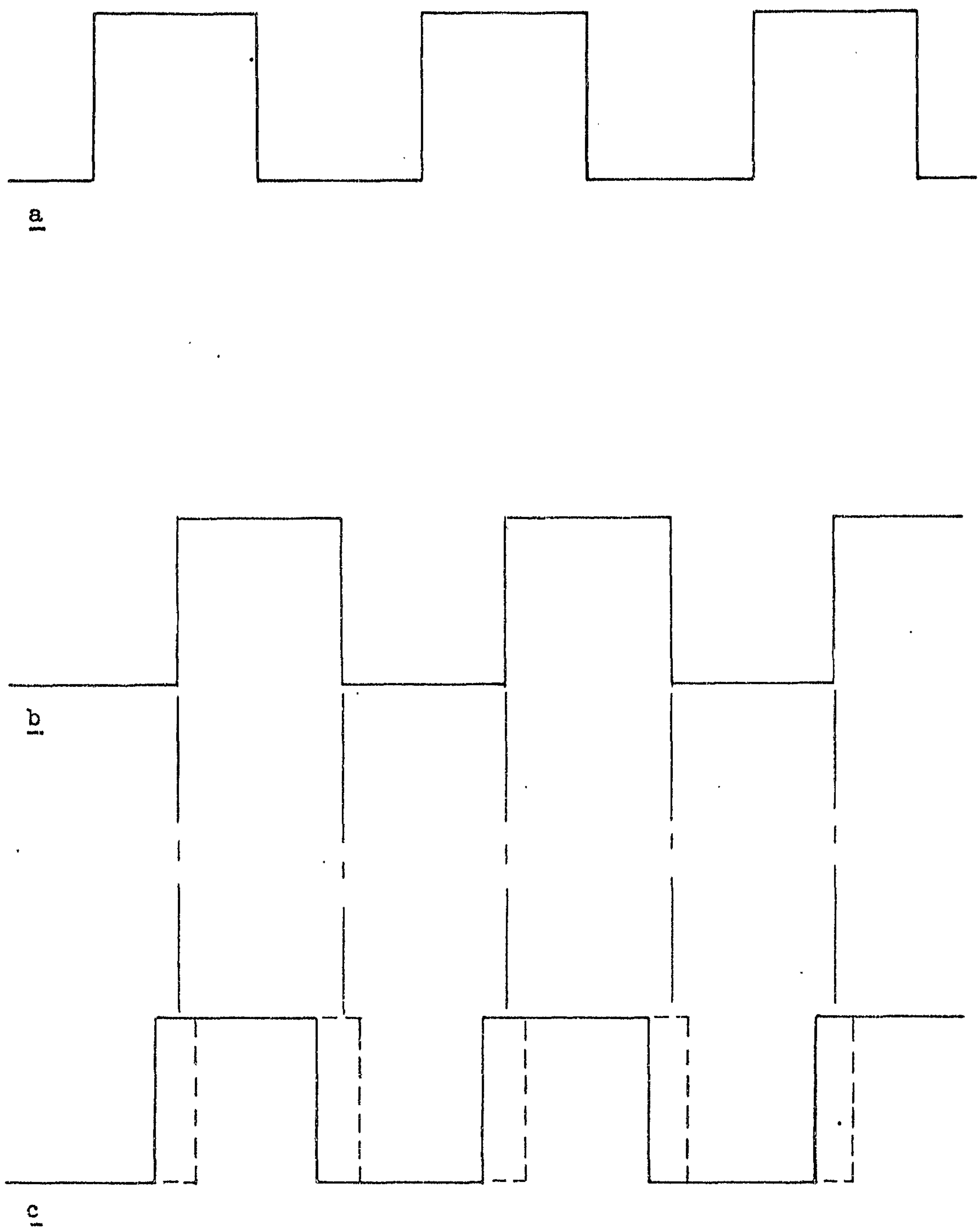


Fig. 16. Waveforms Derived from Potentiometers. Fig. a in Phase with Forcing Function; Fig. b is Lagging Forcing Function by 90°; Fig. c is Derived from Motion of Rocket Motor.

In Fig. 16, a, b, and c are the waveforms. The equation above thus means that the signal to decrease the speed of the motor occurs when a and b and not c are present, or when not a and not b and c are present.

The mechanization of the logic, and operational amplifier, is shown in Fig. 17.

SAMPLER CONVERSION GAIN

At 10 cps (62.8 rad/sec) one radian corresponds to $\frac{1}{62.8}$ second. The pulses out of the detection circuit have amplitudes of 25 volts. Therefore, a 1-radian error would produce a $\frac{25}{62.8}$ volt-second pulse.

If a constant error of one radian exists, and if this error is operated on by one unit impulse function, a pulse is produced whose area is one radian-second. Since the sampling circuit produces a pulse whose area is $\frac{25}{62.8}$ volt-seconds for a 1-radian error, the conversion gain is

$$K_{c\,(62.8)} = \frac{\frac{25}{62.8}\text{ volt-seconds}}{1\text{ radian-second}} = 0.4\ \frac{\text{volts}}{\text{radian}} \quad . \tag{76}$$

At 16 cps (102 rad/sec) a 1-radian error produces pulses whose areas are $\frac{25}{102}$ volt-seconds. Therefore the conversion gain is

$$K_{c\,(102)} = .245 \quad . \tag{77}$$

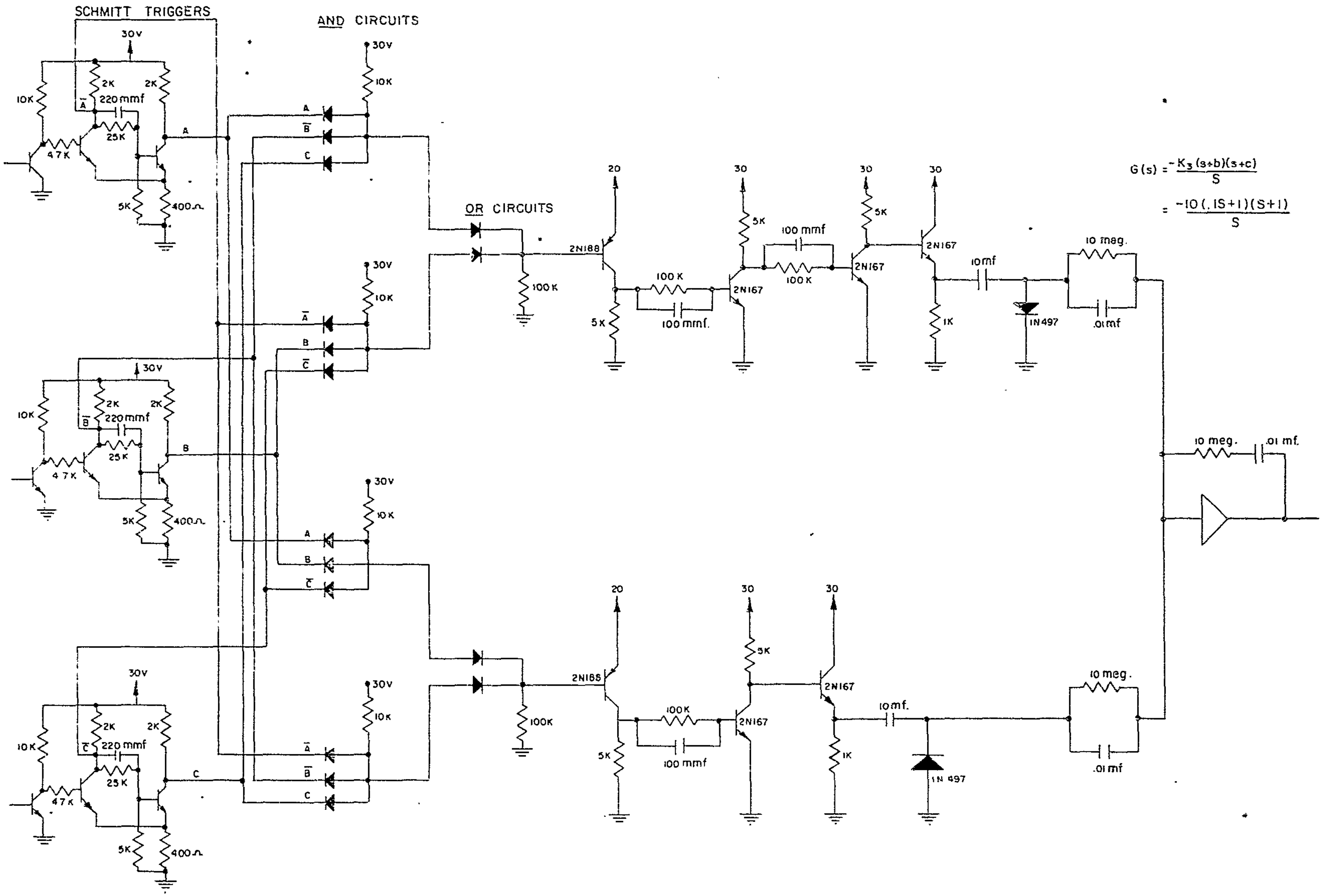


FIG. 17. Circuit to Produce an Error Signal for Every Zero Crossing of the Sinusoidal Input, and the Operational Amplifier Integration and Compensation Circuit.

CONTROL SYSTEM ANALYSIS

ANALYSIS AT 10 CPS AND AT 16 CPS

The system described does not have parameters that are constant in time. As was seen in the previous section, the gain of the system at 10 cps is 1.6 times as great as the gain at 16 cps. Also, the error is sampled and the characteristics of the sampled system are dependent upon the sampling frequency. The sampling rate of this system changes from 20 samples per sec at the beginning of motor burning to 32 samples per sec at the end of burning.

Therefore the system will be analyzed at the two extremes to determine its performance. It is to be observed that the changes in the gain and the sampling frequency occur very slowly, compared to the time constants of the system.

z-TRANSFORM[5]

To analyze the sampled-data control system, the z-transform will be used. The z-transform is defined as

(78) $$z = e^{sT} .$$

The z-transform describes the behavior of a signal at the sampling instant. Consider the system in Fig. 18.

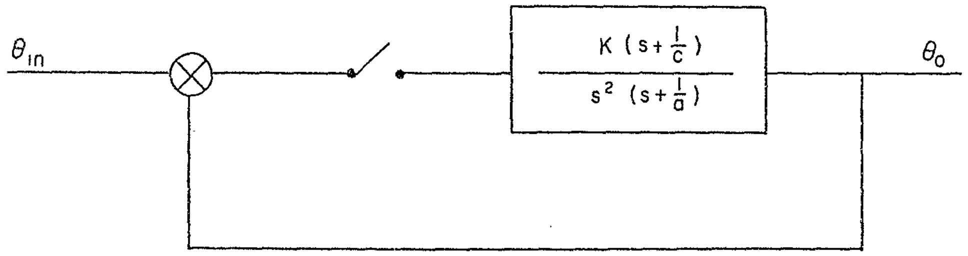


Fig. 18. Sampled-Data System to be Analyzed.

The z-transform of the open-loop transfer function can be found by expanding the function into a partial fraction expansion, and then by means of a table of transform pairs, substituting the z-transform for the s-transform of each of the terms of the expansion.

The expansion of the open loop transfer function of Fig. 18 is

$$(79) \qquad G(s) = \frac{K\left(s + \frac{1}{c}\right)}{s^2\left(s + \frac{1}{a}\right)} = Ka^2\left[\frac{\frac{1}{ac}}{s^2} + \frac{\frac{1}{a} - \frac{1}{c}}{s} + \frac{\frac{1}{c} - \frac{1}{a}}{s + \frac{1}{a}}\right] .$$

The z-transform is then[5]

$$(80) \qquad G(z) = Ka^2\left\{\frac{\frac{T}{ac}z}{(z-1)^2} + \frac{\left(\frac{1}{a} - \frac{1}{c}\right)z}{(z-1)} + \frac{\left(\frac{1}{c} - \frac{1}{a}\right)z}{\left(z - e^{T/a}\right)}\right\}$$

where T is the period between samples.

When T = .05 second—that is when the sampling rate is 20 samples per sec—and given $\frac{1}{a} = 11$, G(z) is then given by

$$(81) \qquad G(z)_{(20)} = \frac{Kz\left[z\left(\frac{.13}{c} + 4.62\right) + \left(\frac{.1}{c} - 4.62\right)\right]}{121\,(z-1)^2\,(z - .58)} .$$

It can be seen that, for $\frac{1}{c} = 1$, the zero comes very close to the pole at +1. This is analogous to placing a zero very close to the origin in the s-plane when the effects of a pole at the origin is to be offset.

The z-transform when $\frac{1}{c} = 1$ is then

$$(82) \qquad G(z)_{(20)} = \frac{Kz\,(4.75)(z - .956)}{121\,(z-1)^2\,(z - .58)} .$$

When $T = .03$ second—that is, when the sampling rate is 32 samples per second—and given $\frac{1}{a} = 11$ and $\frac{1}{c} = 1$, the z-transform is then

$$G(z)_{(32)} = \frac{Kz\ (3.13)(z - .974)}{121\ (z - 1)^2\ (z - .72)} \tag{83}$$

ROOT LOCUS ANALYSIS IN THE z-PLANE

The imaginary axis of the s-plane maps into the unit circle in the z-plane. The area to the left of the imaginary axis in the s-plane maps into the area internal to the unit circle in the z-plane. Therefore, for absolute stability, all the poles of the z-transform must lie within the unit circle.

The constant damping lines, $\xi = \cos \theta$, of the s-plane map into heart-shaped curves in the z-plane. The derivation of the equation for the constant damping curves in the z-plane is shown in Appendix B.

It is then possible to plot the root locus in the z-plane and thereby determine the gain of the system to produce a characteristic transient response.

Root-locus plots of the open-loop z-transforms given in Equations 82 and 83 are shown in Figs. 19 and 20; the unit circle and the constant damping ratio plots for $\xi = 0.2$ and $\xi = 0.4$ are also shown.

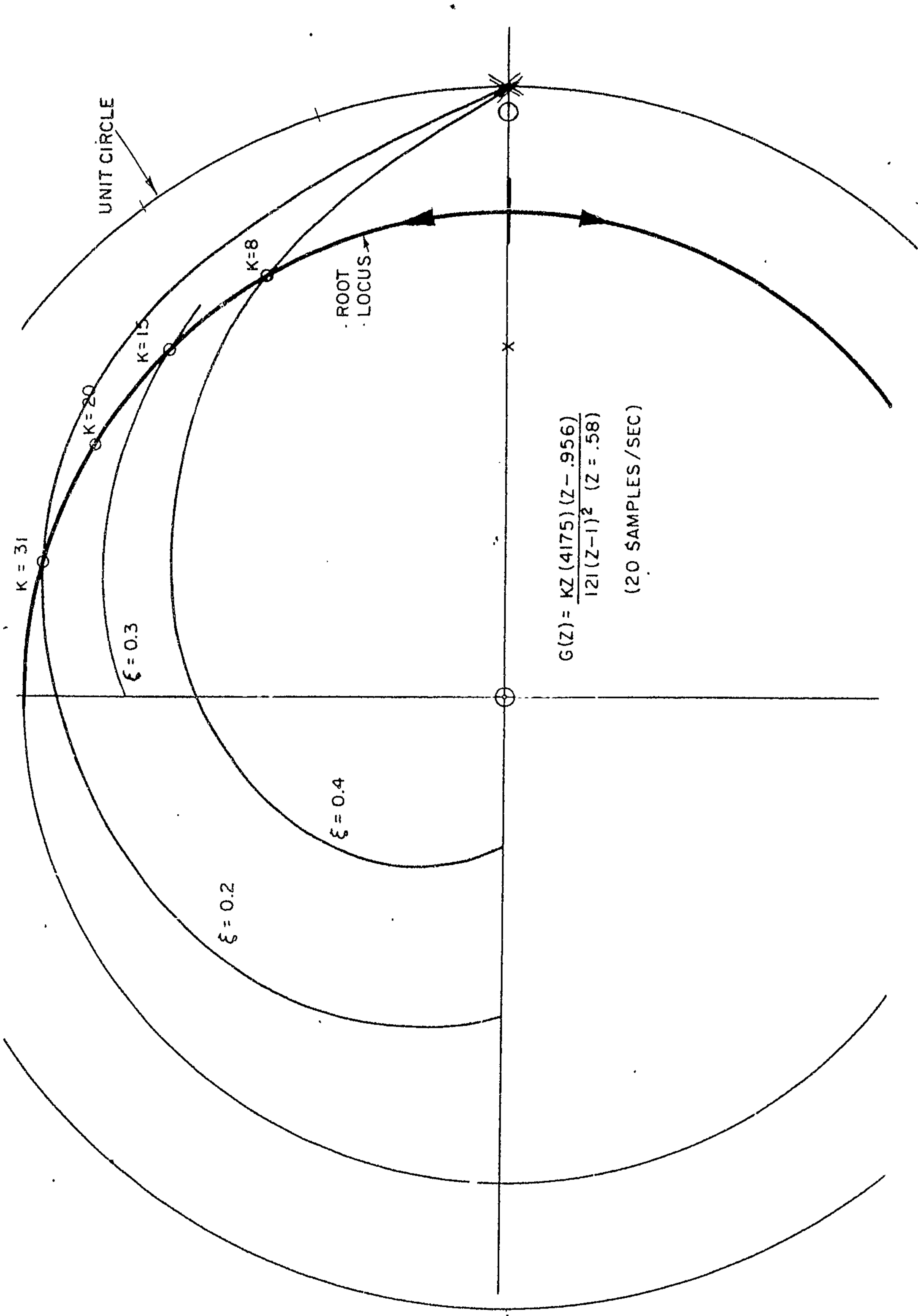


Fig. 19. Root Locus Plot for System Before Motor Burning.

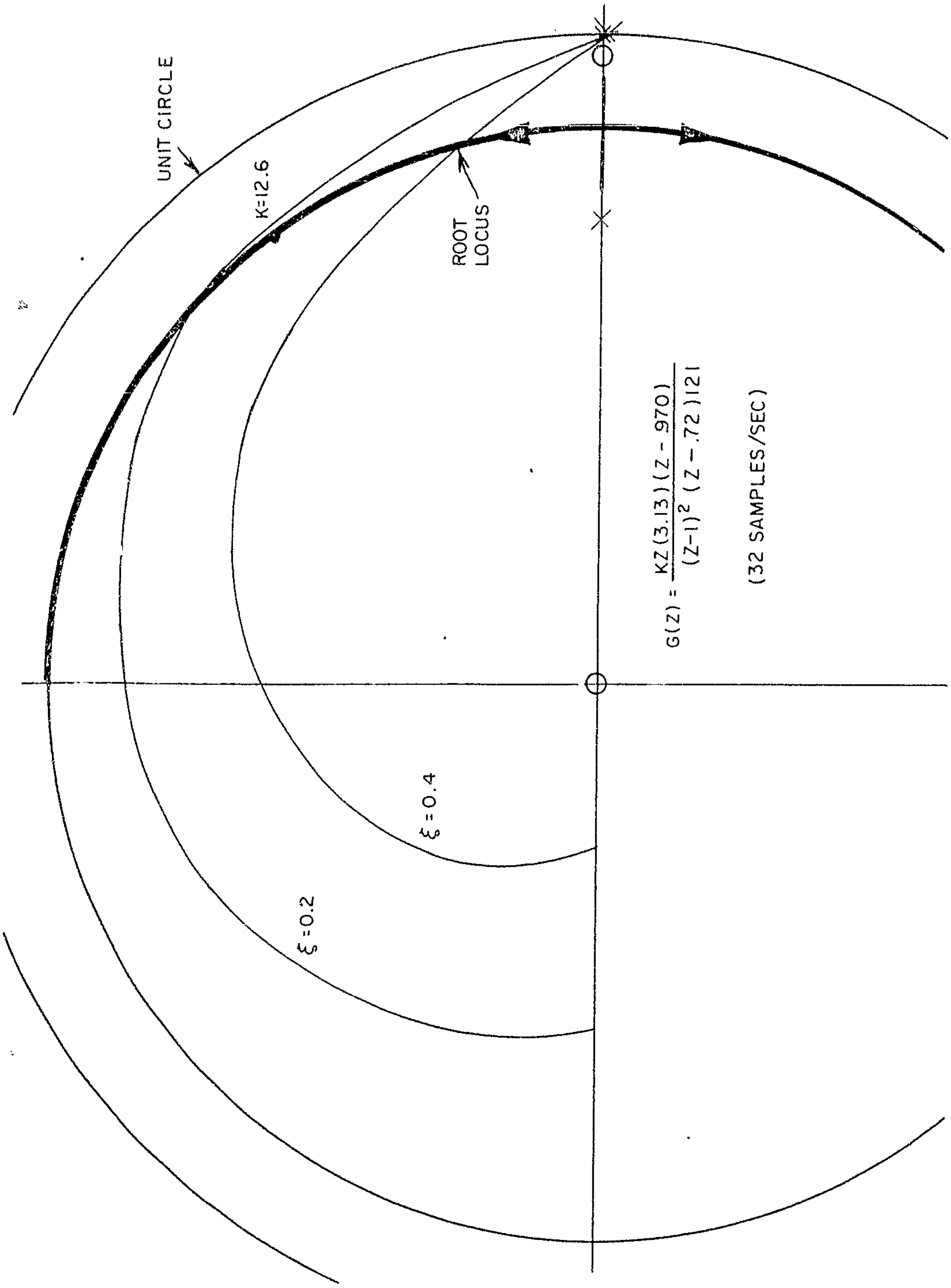


Fig. 20. Root Locus Plot for System After Motor Burning.

GAIN DETERMINATION

To choose the maximum gain for the system to be consistent with good stability, the choice should be made for the case where the system oscillates at 20 cps. The gain of the system is largest at that time, and because the system is being sampled at the lowest rate, it would have the greatest tendency for instability.

The open-loop transfer function of the system before burning is given by Equation 82. The gain at any point of the root locus can be found by[6]

$$K = \frac{1}{|G(z)|} = \frac{121\ (z - 1)^2(z - .58)}{4.75\ z(z - .956)} \quad . \tag{84}$$

The values for the terms in the preceding equation are taken from the graph of the root-locus plot and evaluated for several points along the locus. These are shown on the root-locus plot.

Recall that it was required to have an overshoot in the system response to a step mass change, and that the error should settle to the steady-state value as quickly as possible.

A damping ratio of 0.5 is the value generally chosen to meet the above requirements when the input to a system is a step function. However, as was shown, a step mass change does not produce a step phase change, but rather an exponentially rising function. Therefore, a damping ratio lower than 0.5 is chosen. For the system before burning, a value of $\xi = .25$ was selected, and as will be shown was a satisfactory choice.

The gain for the system with $\xi = .25$ is found by determining the gain at the point where the root locus intersects the constant $\xi = .25$ curve, and is

$$(85) \qquad K_{10} = \frac{121}{4.75} \frac{(z-1)^2(z-.58)}{z\,(z-.956)} = \frac{121}{4.75} \frac{(.88)^2(.67)}{(.85)\,(.77)} = 20.2 \quad .$$

The gain of the system when it is oscillating at 16 cps is

$$(86) \qquad K_{16} = \frac{20.2}{1.6} = 12.6 \quad .$$

STEADY-STATE ERROR

The steady-state error at the instants of sampling, as given by Jury[7] is

$$(87) \qquad e_{ss}(nT) = \lim_{z \to 1} \frac{z-1}{z} R(z) \frac{1}{1+G(z)}$$

where R(z) is the z-transform of the input signal to the system, and G(z) is the open-loop transfer function.

From Fig. 13 it is seen that the rate of change of frequency at the beginning of motor burning is .35 rad/sec^2, and at the end of motor burning is 1.0 rad/sec^2.

The input to the system is then given by

$$(88) \qquad \theta(t) = \frac{kt^2}{2}$$

which is $\theta(t) = \frac{.35t^2}{2}$ for the beginning-of-burning case, and is $\theta(t) = \frac{t^2}{2}$ for the end-of-burning case. The z-transform for $\frac{kt^2}{2}$ given by Truxel[5] is $\frac{1}{2}\,kT^2\,\frac{z(z+1)}{(z-1)^3}$, where T is the sampling period.

The steady-state error for the beginning-of-burning case (system oscillating at 10 cps) is then

$$(89) \qquad e_{ss}(nT)_{10} = \lim_{z \to 1} \frac{z-1}{z} (.35)\frac{1}{2} (.05)^2 \frac{z(z+1)}{(z-1)^3}$$

$$\left[\frac{1}{1 + \frac{(20.2)z}{121} \frac{(4.75)(z-.956)}{(z-1)^2(z-.58)}} \right] = .01 \text{ radian}$$

$$= .57 \text{ degrees.}$$

The steady-state error for the end-of-burning case (system oscillating at 16 cps) is then

$$(90) \qquad e_{ss}(nT)_{16} = \lim_{z \to 1} \frac{z-1}{z} \frac{1}{2} (.031)^2 \frac{z(z+1)}{(z-1)^3}$$

$$\left[\frac{1}{1 + \frac{12.6}{121} \frac{(3.13)z\ (z-.974)}{(z-1)^2(z-.72)}} \right] = .026 \text{ radian}$$

$$= 1.5 \text{ degrees.}$$

If a damping ratio of 0.2 (twice that estimated for the end-of-burning case) is used in Equation (6), it is seen that a steady-state error of 1.5° is adequate for the required system accuracy.

RESPONSE TO A SUDDEN MASS CHANGE

Given the open-loop z-transfer function of the control system for the before-burning case

$$(91) \qquad G(z) = \frac{20(4.75\ z)(z-.956)}{121\ (z-1)^2(z-.58)}$$

the error function is given by

$$(92) \qquad E(z) = \frac{R(z)}{1+G(z)} = \frac{R(z)}{1 + \frac{20(4.75\ z)(z-.956)}{121\ (z-1)^2(z-.58)}}.$$

where $R(z)$ is the transform of the input. For a step input $R(z) = \frac{z}{z-1}$.

Then

$$E(z)_{step} = \frac{z^3 - 1.58\,z^2 + .58\,z}{z^3 - 1.80\,z^2 + 1.41\,z - .58} . \tag{93}$$

If $E(z)$ is expanded in a power series, the coefficients of z^{-n} are the value of $C(nT)$, the time series.[7]

The power expansion of $E(z)$ for the step input is

$$E(z) = 1 + \frac{.22}{z} - \frac{.43}{z^2} - \frac{.50}{z^3} - \frac{.16}{z^4} + \frac{.17}{z^5} + \frac{.22}{z^6} + \frac{.07}{z^7} - \frac{.08}{z^8} - \frac{.11}{z^9} - \frac{.05}{z^{10}} + \frac{.02}{z^{11}} + \frac{.05}{z^{12}} + \frac{.03}{z^{13}} - \frac{.01}{z^{14}} - \frac{.03}{z^{15}} - \frac{.02}{z^{16}} + \cdots\cdots . \tag{94}$$

But the input to the control system for a sudden change of mass is an exponential function of the form

$$v(t) = 1 - e^{-at} \tag{95}$$

$$v(s) = \frac{1}{s} - \frac{1}{s+a} \qquad \text{and} \tag{96}$$

$$v(z) = \frac{z}{z-1} - \frac{z}{z - e^{aT}} = \left(\frac{z}{z-1}\right)\left(\frac{1 - e^{-aT}}{z - e^{-aT}}\right) . \tag{97}$$

The time constant, $\frac{1}{a}$, was shown to be equal to $\frac{1}{\xi\omega}$. When $\omega = 62.8$ rad/sec, $\xi = .15$ and $\frac{1}{a} = \frac{1}{62.8(.15)}$, or approximately 0.1 second.

Then

$$v(z) = \left(\frac{z}{z - 1}\right)\left(\frac{.39}{z - .61}\right) . \tag{98}$$

The error function with an exponential input is, therefore, the error function of the step input multiplied by $\left(\frac{.39}{z - .61}\right)$. The power expansion is

$$E(z)_{\text{Exp.}} = \frac{.39}{z} + \frac{.32}{z^2} + \frac{.03}{z^3} - \frac{.18}{z^4} - \frac{.17}{z^5} - \frac{.04}{z^6} + \frac{.07}{z^7} + \frac{.07}{z^8} + \frac{.01}{z^9} \tag{99}$$

$$- \frac{.05}{z^{10}} \; \frac{.03}{z^{11}} - \frac{.01}{z^{12}} + - - - - - - - .$$

The error functions for the end-of-burning case are found in a like manner. The power expansion for the step input is given below.

$$E(z)_{\text{step}} = \frac{z^3 - 1.72\, z^2 + .72\, z}{z^3 - 2.40\, z^2 + 2.12\, z - .72} \tag{100}$$

$$= 1 + \frac{.68}{z} + \frac{.23}{z^2} - \frac{.17}{z^3} - \frac{.41}{z^4} - \frac{.45}{z^5} - \frac{.33}{z^6} - \frac{.14}{z^7} + \frac{.04}{z^8}$$

$$+ \frac{.16}{z^9} + \frac{.20}{z^{10}} + \frac{.17}{z^{11}} + \frac{.11}{z^{12}} + \frac{.04}{z^{13}} - \frac{.01}{z^{14}} - \frac{.02}{z^{15}} + \frac{0}{z^{16}}$$

$$+ \frac{.03}{z^{17}} + \frac{.06}{z^{18}} + \frac{.08}{z^{19}} + \frac{.08}{z^{20}} + \frac{.06}{z^{21}} + \frac{.03}{z^{22}} + - - - - - - .$$

For the end-of-burning case $\frac{1}{\zeta\, \omega}$ is approximately 0.1 second, and time between samples is $\frac{1}{32}$ second—that is, T = .031 second. Therefore, the step error function is multiplied by $\left(\frac{.27}{z - .72}\right)$ to produce the exponential response. The exponential power expansion is

$$(101)\quad E(z)_{Exp.} = \frac{.27}{z} + \frac{.38}{z^2} + \frac{.35}{z^3} + \frac{.21}{z^4} + \frac{.04}{z^5} - \frac{.09}{z^6} - \frac{.16}{z^7} - \frac{.15}{z^8}$$

$$- \frac{.10}{z^9} - \frac{.03}{z^{10}} + \frac{.04}{z^{11}} + \frac{.07}{z^{12}} + \frac{.08}{z^{13}} + \frac{.07}{z^{14}} + \frac{.05}{z^{15}} + \frac{.03}{z^{16}}$$

$$+ \frac{.02}{z^{17}} + \frac{.02}{z^{18}} + \frac{.03}{z^{19}} + \frac{.05}{z^{20}} + \frac{.05}{z^{21}} + \frac{.05}{z^{22}} + - - - - .$$

From the power expansions of the error functions the output can be plotted. The output is found by subtracting the error from the input. The transient response curves for the four cases given are shown in Fig. 21.

From Fig. 20 it is seen that the overshoot for the exponential inputs is about 10 percent above the final value. The choice of damping ratio appears to be a good one.

FREQUENCY RESPONSE

A knowledge of the frequency response of the system is desirable in order that the effects of unwanted frequency inputs to the system can be determined.

The value of $G(z)$ for various angular frequencies can be found from the z-plane in the following manner:

$$(102)\qquad z = e^{sT} = e^{j\omega T} .$$

Locate z corresponding to a frequency, ω, on the unit circle. $G(j\omega T)$ can then be determined graphically. The factors in the numerator of $G(j\omega T)$ are the vectors from the zero to the value $z = e^{j\omega T}$; the factors in the denominator are the vectors from the poles to the value $z = e^{j\omega T}$.

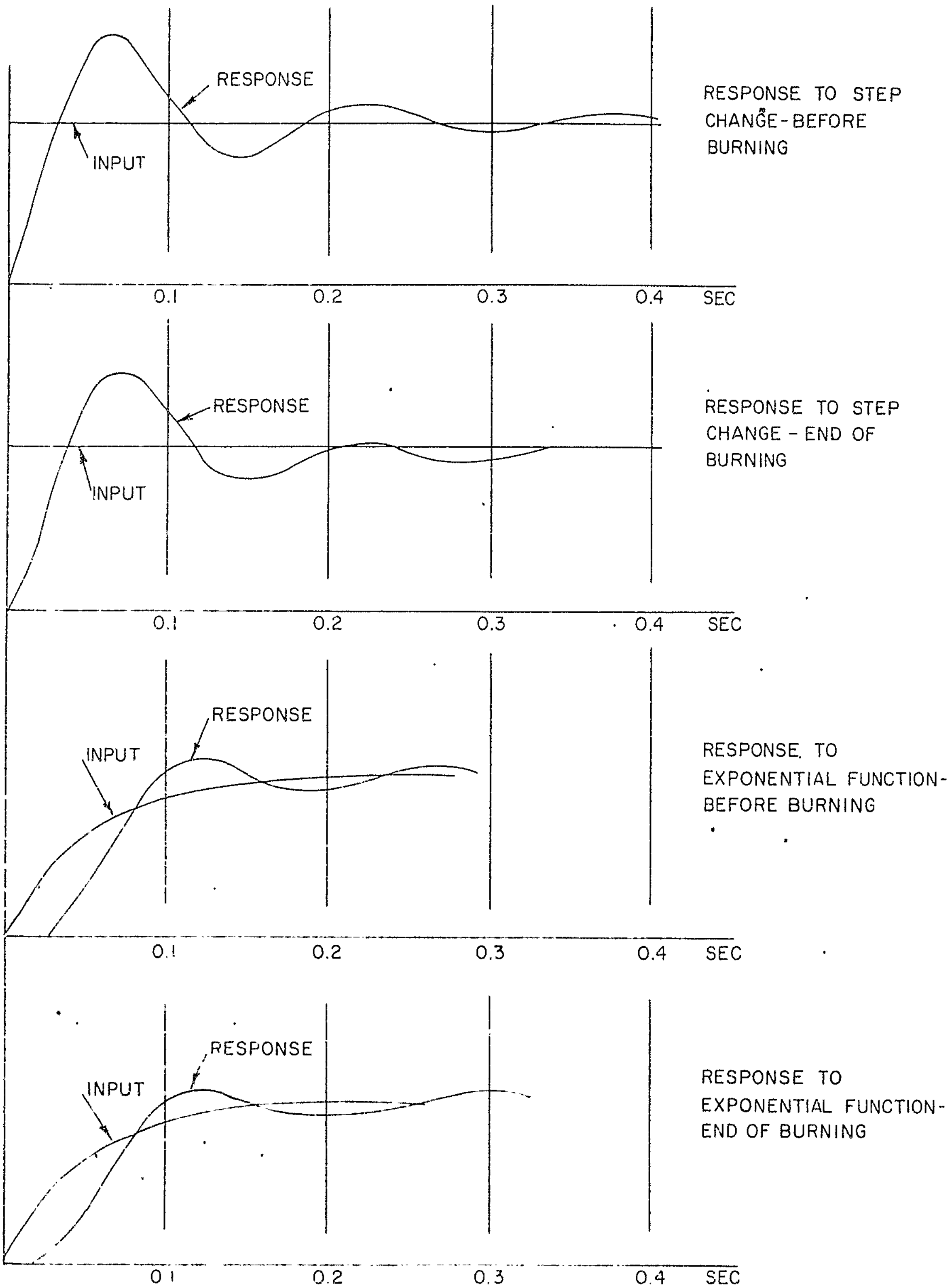


Fig. 21. Transient Response of Control System.

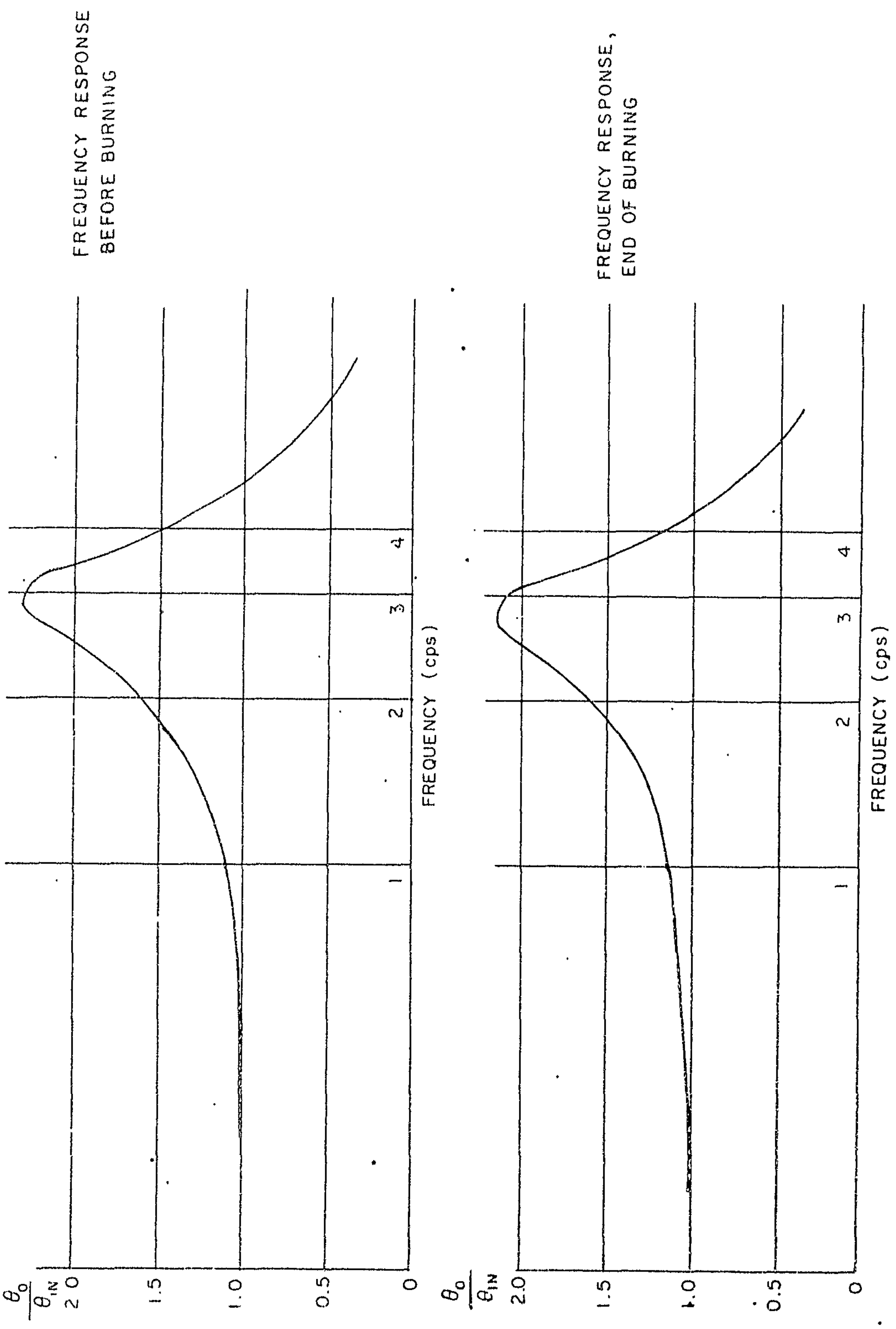


Fig. 22. Frequency Response of Control System.

The ratio of the output to the input is given by

$$\frac{\theta_o}{\theta_{in}} = \frac{G(z)}{1 + G(z)} \quad . \tag{103}$$

The expression $1 + G(z)$ can be determined by adding one to the vector $G(z)$. $\frac{\theta_o}{\theta_{in}}$ is then found by the division of vector quantities.

The frequency response found in this manner for the beginning-of-burning case and for the end-of-burning case is shown in Fig. 22.

An examination of the frequency response shows that the gain in both cases at 1 cps is very slightly greater than unity, and therefore the double rc filter with an attenuation of 42 db is more than adequate in providing the 31 db attenuation required for the specified system accuracy.

It should also be pointed out that if an unwanted signal of 3 cps is present, the rc filter provides only 23 db attenuation, while the system has a gain of 7 db (2.25 voltage gain) to give a net attenuation of only 16 db.

REACTION TORQUE

The shaker system vibrating in a vertical direction exerts a reaction torque on the eccentric mass of the shaker motor.

Given the system below (Fig. 23),

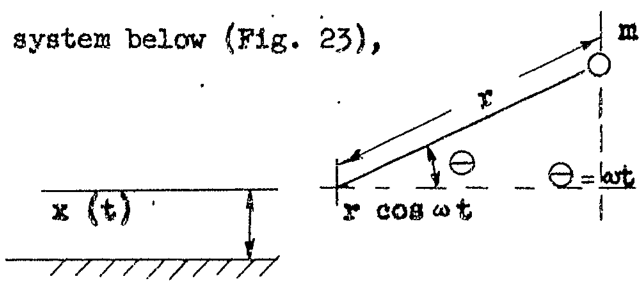


Fig. 23. Rotating System with the Center of Rotation Moving Sinusoidally in the Vertical Direction.

the vertical oscillation when the system is vibrating at the natural frequency is given by Equation (7),

$$x(t) = \frac{mr\,\omega^2}{2\,\xi\,K} \sin\left(\omega t - \frac{\pi}{2}\right) \quad .$$

The force on the mass is

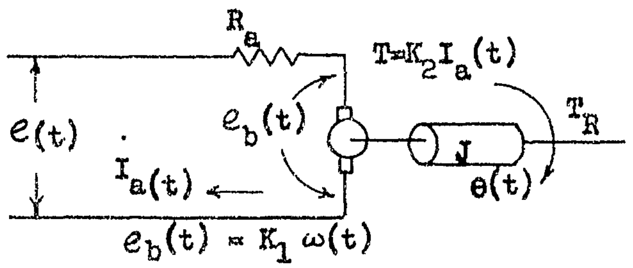
(104) $$f(t) = m\ddot{x} \quad .$$

The reaction torque is then

(105) $$T_R = f(t)\, r \cos \omega t = \frac{(mr)^2\,\omega^4}{2\,\xi\,K} \sin\left(\omega t - \frac{\pi}{2}\right) \cos \omega t$$

$$= \frac{(mr)^2\,\omega^4}{2\,\xi\,K} \left(\frac{1 + \cos 2\,\omega t}{2}\right) \quad .$$

In a manner similar to that shown in Brown and Campbell,[3] the transfer function of the dc motor will be derived with the reaction torque T_R as part of the load. The schematic of the motor and load is shown in Fig. 24.



$e(t)$ = applied voltage
R_a = armature resistance
T_R = reaction torque load
J = armature and load moment of inertia
$\theta(t)$ = output shaft position
e_b = counter emf

Fig. 24. Schematic Diagram of dc Motor and Load.

From the equivalent circuit and the equalities given, we may write the following equations:

(106) $$E(s) = R_a\, I_a(s) + E_b(s)$$

(107) $$E_b(s) = K_1\, s\, \Theta(s)$$

(108) $$T(s) = K_2 \, I_a(s) \; .$$

(109) $$T(s) = Js^2 \, \Theta(s) + T_R(s) \quad .$$

By substitution, the following equation is found:

(110) $$E(s) - \frac{R_a}{K_2} T_R(s) = K_1 \, \Theta(s) \, s \left[\frac{R_a J}{K_1 K_2} s + 1 \right] \quad .$$

The above equation shows that the reaction torque can be considered as an input voltage subtracted from the normal input to the motor as indicated below (Fig. 25):

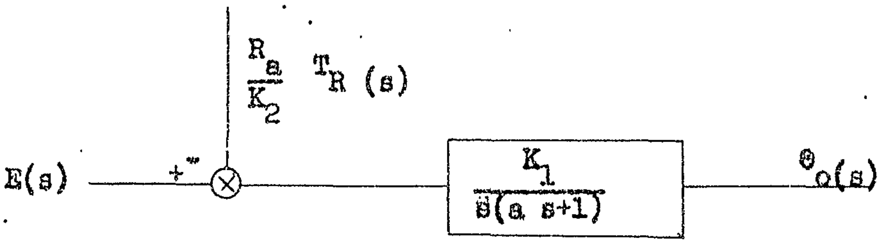


Fig. 25. Equivalent System of dc Motor with Reaction Torque.

The complete control system is shown in Fig. 26.

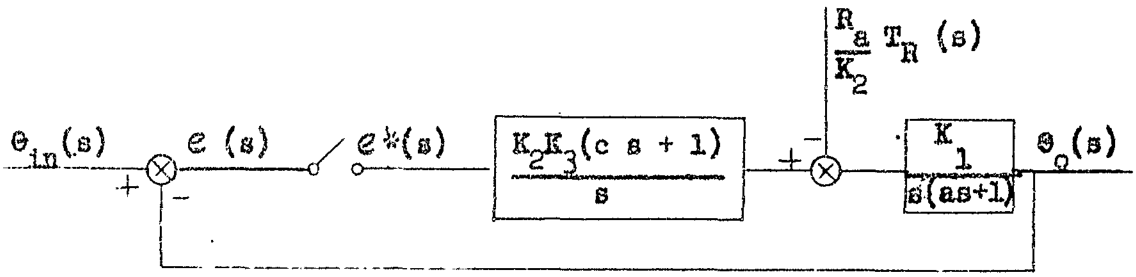


Fig. 26. Control System with Reactive Torque Input.

For $\Theta_{in}(s) = 0$, the z-transform of the output for the system in Fig. 26 is

(111) $$\theta_o(z) = -\frac{RG_2(z)}{1 + G_1G_2(z)}$$

where

(112) $$G_1(s) = \frac{K_2K_3(c\ s + 1)}{s}\ ,$$

(113) $$G_2(s) = \frac{K_1}{s\ (a\ s + 1)}\ ,$$

and

(114) $$R(s) = \frac{R_a}{K_2}\ T_R(s)\ .$$

(115) $$R(t) = \frac{R_a}{K_2}\ T_R(t) = \frac{R_a\ (mr)^2\ \omega^4}{K_2\ 2\ \xi\ K}\left(\frac{1 + \cos 2\ \omega t}{2}\right)\ .$$

Let

(116) $$\frac{R_a\ (mr)^2\ \omega^4}{K_2\ 4\ \xi\ K} = B\ .$$

Then

(117) $$RG_2(s) = B\left(\frac{1}{s} + \frac{\frac{s}{4\ \omega^2}}{1 + s^2/4\omega^2}\right)\frac{K_1}{s\ (a\ s + 1)}$$

and the time function as given by Nixon[2] is

(118) $$RG_2(t) = BK_1\left[a\left(e^{-\frac{t}{a}} + \frac{t}{a} - 1\right) + \frac{1}{4\ \omega^2}\left(\frac{a\ 4\ \omega^2}{1 + a^2\ 4\omega^2}\ e^{-\frac{t}{a}}\right.\right.$$
$$\left.\left. + \frac{2\omega\ \sin\ (2\omega t - \psi)}{(1 + a^2\ 4\omega^2)^{1/2}}\right)\right]\ .$$

The steady state value is

$$(119) \qquad RG_2(t)_{ss} = \lim_{t \to \infty} RG_2(t) = BK_1\left[t + \frac{2\,\omega \sin(2\omega t - \psi)}{4\omega^2\,(1 + 4a^2\omega^2)^{1/2}}\right]$$

$$= BK_1\left[t + \frac{\sin(2\omega t - \psi)}{2\omega\,(1 + 4a^2\omega^2)^{1/2}}\right]$$

where

$$\psi = \tan^{-1} 2\,a\,\omega \quad .$$

Sin $(2\omega t - \psi)$ can be written as

$$(120) \qquad \sin(2\omega t - \psi) = \frac{1}{\sqrt{4a^2\,\omega^2 + 1}} \sin 2\omega t - \frac{2a\omega}{\sqrt{4a^2\,\omega^2 + 1}} \cos 2\omega t \quad .$$

Then

$$(121) \qquad RG_2(t)_{ss} = BK_1\left[t + \frac{\sin 2\omega t}{(1 + 4a\,\omega)}\,2\omega - \frac{a \cos 2\omega t}{(1 + 4a^2\omega^2)}\right] \quad .$$

and the z-transform from Jury[7] is

$$(122) \qquad RG_2(z)_{ss} = BK_1\left[\frac{T\,z}{(z - 1)^2} + \frac{z \sin 2\omega t}{(z^2 - 2z \cos 2\omega T + 1)(1 + 4a^2\omega^2)}\,2\omega\right.$$

$$\left. - \frac{az\,(z - \cos 2\omega t)}{(z^2 - 2z \cos 2\omega T + 1)(1 + 4a^2\omega^2)}\right] \quad .$$

But

$$(123) \qquad T = \frac{1}{2f} = \frac{\pi}{\omega} \quad .$$

Therefore,

$$(124) \qquad RG_2(z)_{ss} = BK_1\left[\frac{T\,z}{(z - 1)^2} - \frac{az\,(z - 1)}{(z^2 - 2z + 1)(1 + 4a^2\omega^2)}\right]$$

$$= BK_1\left[\frac{T\,z}{(z - 1)^2} - \frac{az}{(z - 1)(1 + 4a^2\,\omega^2)}\right] \quad .$$

It is to be noticed that in sampling a sine function at the frequency of the function, the result appears as a dc value. The function $\frac{z}{z - 1}$ is the transform of a step function.

For the before-burning case

$$G_1G_2(z) = G(z) = \frac{K\, z\, (4.75)(z - .956)}{121\, (z - 1)^2(z - .58)} \tag{125}$$

$$\theta_o(z) = \frac{RG_2(z)}{1 + G_1G_2(z)} \quad . \tag{126}$$

The steady-state error of $\theta_o(z)$ from the final value theorem is

$$\theta_o(z)_{ss} = \lim_{z\to 1} \frac{z - 1}{z} \; \frac{RG_2(z)}{1 + G_1G_2(z)} \tag{127}$$

$$= \lim_{z\to 1} \frac{z - 1}{z} \; \frac{BK_1\left[\frac{T\, z}{(z - 1)^2} - \frac{az}{(z - 1)(1 + 4a^2\omega^2)}\right]}{1 + \frac{Kz\,(4.75)(z - .956)}{121\,(z - 1)^2(z - .58)}}$$

$$= 0 \quad .$$

The after-burning case, having a transfer function of the same form as the before-burning case, produces the same results.

CONCLUSION

The system analysis indicates that the objectives of the system can be met with the design described. The control system's steady-state error was determined to be .57° for the beginning-of-burning case and 1.5° for the end-of-burning case; it was shown that if the damping of the system were twice that which was estimated, these steady-state error values would produce a mass error of less than .5 percent.

For a sudden change in mass the system responds with an overshoot of approximately 10 percent, providing an indication of the change, and accomplishing the change so that the new steady-state value is approached in a short period of time.

The frequency response of the system, in conjunction with filters in the feedback loop and in the input to the control system, provides adequate discrimination against unwanted low frequencies, which amplitude-modulate the 10-16 cps carrier. Assumptions are made that other frequencies not generated by the shaker will not have amplitudes significant enough to affect the accuracy and operation of the system. These assumptions should be verified experimentally; if they should prove to be wrong, steps would be required to eliminate or discriminate against these frequencies. The usefulness of the mass determining system is contingent upon the amplitudes of these frequencies being insignificant.

The effect of the vertical oscillation feeding back into the control system via the eccentric load on the dc motor was shown to be zero, and therefore it would offer no problem to the system.

The analytic study demonstrates the feasibility of measuring the mass of a burning rocket motor by determining the natural frequency of a spring-mass system.

An experimental system is being constructed at the U. S. Naval Ordnance Test Station. As soon as this laboratory system is completed, experiments will be conducted to verify the results obtained in this paper. The proof of the scheme, however, can come only when it is applied to a test stand containing a burning rocket motor.

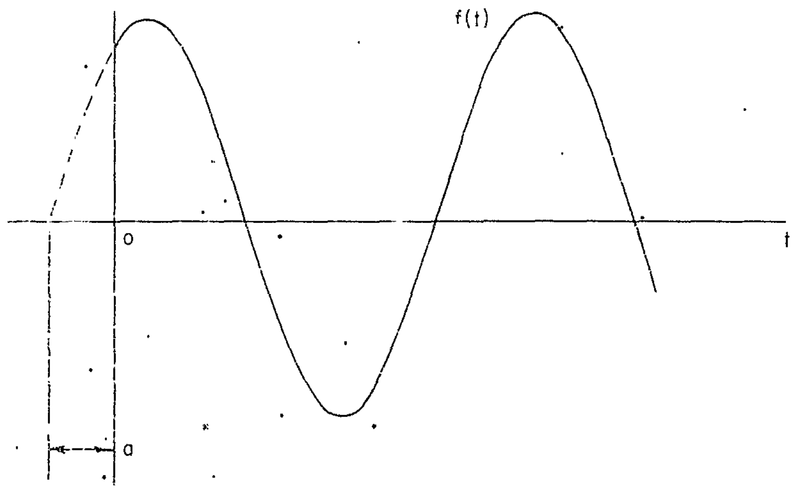
APPENDIX A

Derivation of the Laplace transform of a function that is zero before time zero, and is a sine function delayed by a time "a" for all time greater than zero.

The function described in the title is written as

$$f(t) = \sin \omega(t + a)\, \mu(t) \,, \tag{128}$$

and is pictured as



$$f(t) = mr\,\omega^2 \sin\left[\omega(t + a)\right]\mu(t)$$

Fig. 27. Sine Wave Advanced in Time by "a" Second and Starting at Zero Time.

From Aseltine[1] then we have the identity

$$\mathcal{L}\left[\sin \omega(t+a)\ \mu(t)\right] = e^{as}\int_0^{\infty} \sin \omega t\ \mu(t-a)\ e^{-st}\, dt \ . \tag{129}$$

Therefore,

$$F(s) = e^{as}\int_0^{\infty} \sin \omega t\ \mu(t-a)\ e^{-st}\, dt \tag{130}$$

$$= e^{as}\int_a^{\infty} \sin\ \omega t\, \mu(t-a)\ e^{-st}\ dt \quad .$$

Writing sin ωt in the exponential form, we have

$$F(s) = \frac{e^{as}}{2i}\int_a^{\infty}\left(e^{i\omega t} - e^{-i\omega t}\right)\ e^{-st}\ dt \quad . \tag{131}$$

Integrating gives

$$F(s) = \frac{e^{as}}{2i}\left[\frac{e^{i\omega t\ -\ st}}{i\omega - s} - \frac{e^{-i\omega t\ -\ st}}{-i\omega - s}\right]_a^{\infty} \tag{132}$$

$$= \frac{e^{as}}{2i}\left[-\ \frac{e^{i\omega a\ -\ sa}}{i\omega - s} + \frac{e^{-i\omega a\ -\ sa}}{-i\omega - s}\right] \quad .$$

Multiply the numerator and denominator of both terms by the conjugates of the denominators to give

$$F(s) = \frac{e^{-sa}\ e^{sa}}{2i(\omega^2 + s^2)}\left[i\omega e^{i\omega a} + i\omega e^{-i\omega a} + s e^{i\omega a} - s e^{-i\omega a}\right] \tag{133}$$

Putting the exponential terms into the trigonometric form gives

$$F(s) = \frac{1}{\omega^2 + s^2}\left[\omega \cos\ \omega a + s\ \sin\ \omega a\right] \quad . \tag{134}$$

Factoring ω cos ωa gives

$$F(s) = \frac{\omega\ \cos\ \omega a}{\omega^2 + s^2}\left(\frac{s\ \tan\ \omega a}{\omega} + 1\right) \quad . \tag{135}$$

APPENDIX B

Mapping the Damping Ratio, ξ, into the z-Plane.

The z-transform is defined as

(136) $$z = e^{sT} .$$

Let

(137) $$z = Re^{i\psi}$$

and

(138) $$s = re^{i\theta} .$$

Putting Equation (136) into the logarithmic form gives

(139) $$\log R + i\psi = Tre^{i\theta} .$$

Putting the term on the left side of Equation (4) into the polar form gives

(140) $$\sqrt{\log^2 R + \psi^2}\; e^{i \tan^{-1} \frac{\psi}{\log R}} = Tre^{i\theta} .$$

Therefore

(141) $$\tan \theta = \frac{\psi}{\log R} .$$

Now

(142) $$\xi = - \cos \theta .$$

Therefore

(143) $$\tan \theta = \frac{-\xi}{\sqrt{1 - \xi^2}} .$$

Substituting $\frac{-\xi}{1 - \xi^2}$ for tan θ in Equation (141) gives

(144) $$\log R = \frac{-\xi}{\sqrt{1 - \xi^2}} \psi$$

Taking the anti-log gives

$$R = e^{-\frac{\xi}{\sqrt{1 - \xi^2}}\psi} . \tag{145}$$

Equation (145) defines a curve in the z-plane for fixed values of ξ.

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A control system is designed, analyzed, and shown to meet the requirements of the system with the expected input signals and external disturbances.

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